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SYNTHESIS OF A FINITE, FOUR-TERMINAL NETWORK FROM ITS PRESCRIBED DRIVING-POINT FUNCTIONS AND TRANSFER FUNCTION

By Charles M:son Gewertz[1]

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[1] *Containing the principal results of a research submitted for a Doctor's degree in the Department of Electrical Engineering at The Massachusetts Institute of Technology.*

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INTRODUCTION

STATEMENT OF THE PROBLEM

In network *analysis* either the network and the applied "impulse"[2] (usually a voltage) are given, and we seek the "response" (usually a current),—or the network and the "response" are given, and we seek the "impulse."

These cases are extensively treated in the literature, usually by applying operational methods.

In network *synthesis* the "response" to a certain "impulse" is given (prescribed), and *the problem is to design a network having such properties that it responds in a prescribed manner to the applied impulse.*

Network synthesis is steadily assuming increased scientific interest and technical importance, especially in view of increasing needs in electrical communication engineering and talking movies, etc.

It is more difficult to handle than network analysis, and, as being a more or less new science, the literature on the subject is rather scarce.

When dealing with more than one pair of terminals, we are forced into the field of non-singular[3] (as well as singular) matrices and determinants whose elements are rational, fractional, algebraic functions of a complex variable.

It is because the literature on this subject is exceptionally scarce, that we in this paper, which is a contribution to our knowledge in network synthesis and mainly built up on a mathematical foundation in terms of complex matrices and determinants, shall have to present so many proofs (having been deduced when needed) in the following.

[2] By "impulse" we mean throughout this paper the applied driving force or steady state excitation.

[3] If non-singular a consideration of the inverse matrix is generally also necessary.

In network synthesis the desired response can be stated[4] either in terms of a current (or voltage) as a function of *time* for a constant "unit" voltage ("unit" direct current) applied, i.e. a voltage (current respectively) which is $= 0$ for $t < 0$ and unity for $t > 0$, as shown in Fig. 1. This is based on Heaviside's (4)[5] method of attack, and the "response function" embodies an infinite time.

The measured current for a constant unit voltage applied is usually called "*indicial admittance*" (a name coined by Carson (6)), while the measured voltage for a unit direct current applied may be called "*indicial impedance.*"

On the other hand, the response can be stated[6] in terms of a current (or voltage) as a function of *frequency* for a steady alternating voltage (current respectively) applied. We then measure

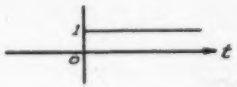


FIG. 1

the response in magnitude and phase at such a time, when the transient has died out to zero.

The response will then be a function of an infinite range of frequencies,—and *it is from three such functions* (referring to an impulse of unit strength) *that we start out when synthesizing our finite four-terminal network.*

The responses measured at the same end of the network as that at which the impulse is applied are the two "*driving-point*" functions, and the response measured at the opposite end to that at which the impulse is applied is the "*transfer*" function.

If using the unit vector representing the steady alternating impulse as reference, and letting it fall along the real axis in a complex plane (as shown in Fig. 2) then it is evident that the vector representing the response (changing with the frequency

[4] Bibliogr. 7, p. 78, pp. 176–181 and p. 291. Bibliogr. 38, p. 93.
[5] For all references by number see Bibliography.
[6] Same reference as footnote 4.

and depending upon the character of the network) at the driving-point cannot have a direction which differs by more than ± 90 degrees from the direction of the impulse vector. *Thus the real part of the response vectors at the driving-point is always* ≥ 0 *for a passive structure,* (i.e. a structure containing "positive" circuit elements[7]), consuming a finite amount of, or zero, power. This power is, for all frequencies, represented by the product of the

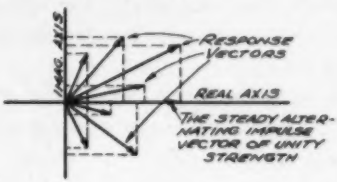


FIG. 2

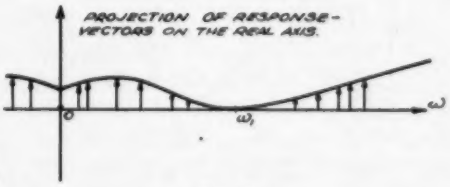


FIG. 3

projection of the response vector upon the real axis and the whole impulse vector falling along this axis.

An angle > ± 90 degrees between the impulse vector and the response vector would simply mean that the network at that particular frequency, instead of consuming, would supply power, which is absurd.

For an infinite range of frequencies we get an infinite range of response vectors, all distributed in the right half of the plane in Fig. 2, and if plotting their projections on the real axis in a plane (see Fig. 3), then we get our Re $f_{(j\omega)}$—ω characteristic,[8] which

[7] Combinations of resistance-, capacitance- and inductance parameters.

[8] "Re" stands for: *"the real part of"*—and "Imag." stands for: *"the imaginary part of"*—throughout.

was successfully used by Brune (38) when synthesizing a finite, dissipative two-terminal network.

When studying Fig. 2 and Fig. 3 together, we notice that the curve in the latter figure does not tell us anything about the shape of a curve representing the projections of the response vectors on the imaginary axis (see Fig. 4 showing[9] our Imag. $f_{(j\omega)}$—ω characteristic).[9]

For instance, when at a frequency ω_1 the real part is = zero, then the response vector may be either 90 degrees leading or 90 degrees lagging.

Or the response vector may be zero at this frequency corresponding to resonance of an *antiresonance component* connected

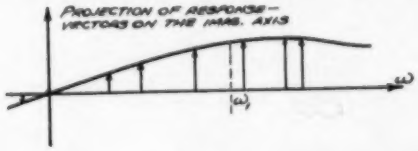


FIG. 4

in series with one of the terminating leads in case the impulse is an alternating unit voltage,—or resonance of a *resonance component* across the terminals in case the impulse is a unit alternating current.

The interesting and invaluable fact for the solution of our problem, however, (pointed out by Lee[10] and Brune[10] and which to a certain extent shall be shown in Parts IX and XI) is this, that when one of the characteristics shown in Fig. 3 and Fig. 4 respectively is known, then the other one can generally be found by the aid of mathematical analysis.

If the imaginary part is known, the real part can be found but for an arbitrary constant, but *in our case we are going to find*

[9] Note, from the shape of the curves in Fig. 3 and Fig. 4 for positive and negative ω:s respectively it becomes clear that Re $f_{(j\omega)}$ — ω characteristic represents an *even* function while Imag. $f_{(j\omega)}$ — ω characteristic represents an *odd* function of ω.

[10] Bibliogr. 37, p. 7. Bibliogr. 38, p. 97.

the imaginary part when the real part is known, and due to the fact that the real part, from which we shall start out, corresponds to a function of a complex variable which cannot have any pole on the boundary, the imaginary part, and thus the λ-function[11] itself, can be uniquely determined.

When studying Fig. 2 it is evident that instead of considering the projections of the response vector on the real and imaginary axis, this vector is also uniquely determined by its amplitude, and its angle to the real axis, (say).

Thus, we can as well deal with two other curves replacing those in Fig. 3 and Fig. 4, i.e. the amplitude-frequency characteristic and the phase-frequency characteristic.

When knowing one of them, the other one can be found. The amplitude-frequency characteristic generates the phase-frequency characteristic uniquely, while the reverse is true but for an arbitrary constant factor.

On page 9 we said that the response vectors in Fig. 2 cannot have an angle, with respect to the impulse vector, $> \pm 90$ degrees.

This restriction, however, refers to our driving-point functions only and does not hold for our transfer function, because this function has no physical meaning in terms of energy-flow, being an alternating current flowing in a short-circuit between the two terminals at one end as a response to an alternating unit voltage of any frequency applied between the two terminals at the opposite end,—or an alternating voltage measured between the two terminals at one end as a response to a unit alternating current of any frequency applied at the opposite end. Thus, the response vectors corresponding to the transfer function may have any angle with respect to the impulse vector, i.e. *their projection on the real axis may be positive, zero or negative*!

Further, in this case all the response vectors can be considered as directed in the opposite direction, corresponding to the reversal of a pair of terminals at either end of the network. In spite of the fact that the response vectors of the transfer function may

[11] As to the definition of λ see Fig. 6, p. 12.

have projections on the real axis of any sign, *the magnitudes of these projections, however, are not arbitrary* but are restricted by a determinant condition as shown in items (c) and (d) of Part III.

It may be added that the transfer function is the most important one, and it is from a curve representing the amplitude-frequency characteristic of this function plus two terminal impedances that we usually proceed in practice.

Then after an ω-function (as a quotient of two polynomials) fitting this curve with the corresponding phase-frequency characteristic are found, the two driving-point functions may be

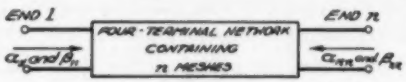


FIG. 5

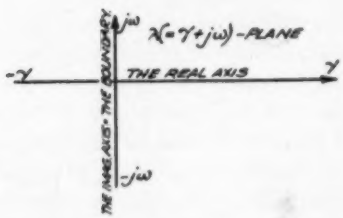


FIG. 6

chosen to give any desired image impedances at the terminals. In addition *certain necessary and sufficient conditions* (to be given in the following Parts) *must be satisfied and then the network can be built by the aid of the general methods outlined in this paper.*

Suppose now that we want to synthesize the four-terminal network shown in Fig. 5.

In our particular problem two pairs of terminals will be considered, and throughout this paper "*end 1*" refers to the pair to the left, while "*end n*" refers to the pair to the right. Subscripts of our λ-functions and potential functions will be in accordance therewith.

All the way we must carefully distinguish between a scalar function $f_{(\lambda)}$ and a function $f_{(j\omega)}$, because in a complex λ-plane (see Fig. 6) the values of $f_{(\lambda)}$ refer to the whole plane, while those of $f_{(j\omega)}$ refer to the "*boundary*" (= imaginary axis) only. As shall be discussed in more detail later, however, *due to Cauchy's*[12] *integral* (*and Poisson's*[13]) *it is possible to reduce our efforts to the boundary only*, i.e. *the necessary and sufficient conditions*, arrived at in the following, *can be stated in terms of conditions satisfied on the boundary* (see Theorem V, p. 88).

Notations: Let us adopt the following notations for our functions mentioned in the title of this paper,—to be used throughout:

α_{11} is the "short-circuit driving-point" *admittance* at end 1, i.e. the alternating *current* measured at end 1 as a response to an alternating unit voltage of any frequency applied at the same end, while the two terminals at end n are shorted.

α_{nn} is the analogous "short-circuit driving-point" *admittance* at end n.

α_{1n} is the "short-circuit transfer" *admittance*, i.e. the alternating *current* measured in the short at either (due to the reciprocal theorem[14]) end as a response to an alternating unit voltage of any frequency applied at the opposite end of the network. Thus α_{1n} also includes and is equal to α_{n1}.

β_{11} is the "open-circuit driving-point" *impedance* at end 1, i.e. the alternating *voltage* measured at end 1 as a response to a unit alternating current of any frequency applied at the same end, while the two terminals at end n are open.

β_{nn} is the analogous "open-circuit driving-point" *impedance* at end n.

[12] Bibliogr. 43 and 44.
[13] Bibliogr. 47, V. II, p. 70.
[14] Bibliogr. 7, p. 33.

β_{1n} is the "open-circuit transfer" *impedance* i.e. the alternating *voltage* measured between the two terminals at either (again due to the reciprocal theorem[14]) end as a response to a unit alternating current of any frequency applied at the opposite end of the network. Thus β_{1n} also includes and is equal to β_{n1}.[15]

Due to the fact that our functions characterize a four-terminal network having a *finite* number of meshes *they must each be a quotient of two rational integral polynomials in* λ, i.e. when considering the α:s (say):

$$\left.\begin{aligned} \alpha_{11} &= \frac{g_{1(\lambda)}}{h_{(\lambda)}} \\ \alpha_{nn} &= \frac{g_{n(\lambda)}}{h_{(\lambda)}} \\ \alpha_{1n} &= \frac{g_{1n(\lambda)}}{h_{(\lambda)}} \end{aligned}\right\} \quad \ldots\ldots (1)^{16}$$

The β:s can be defined in an analogous manner. It is sufficient, however, to define *one* family,[17] because the other one can then be defined in terms of the first one, as shall be shown in Part I.

Further, the sign and character of the coefficients and the location of the roots in the λ-plane of the $g_{(\lambda)}$:s, of their determinant and of $h_{(\lambda)}$ are not arbitrary, as shall be shown in Part II.

In addition, conditions will be imposed upon the real parts of our functions and upon the determinant of the real parts (shown in Part III) as well as upon the residues and the determinant of

[15] Note the α:s and β:s ought to be written $\alpha_{11(\lambda)}$, $\alpha_{nn(\lambda)}$, $\alpha_{1n(\lambda)}$, $\beta_{11(\lambda)}$, $\beta_{nn(\lambda)}$ and $\beta_{1n(\lambda)}$, but for the sake of simplicity the subscript (λ) will be left out throughout.

[16] The roots of the numerator are called "*zeros*" and the roots of the denominator are called "*poles*."

[17] With "*family*" we shall always mean a set of α-functions or a set of β-functions throughout.

the residues at singularities along the boundary, (shown in Part IV).

All these conditions are termed *necessary conditions*, and only if every matrix of functions, besides fulfilling the necessary conditions, also proves to be realizable into a passive network, are these conditions also *sufficient*. Perhaps the necessary conditions will have to be supplemented.

Thus instead of the formulation in the title of this paper our problem could as well have been defined as:

Three rational, fractional functions, supposed to be the two "short-circuit driving-point" admittances and the "short-circuit transfer" admittance, or the two "open-circuit driving-point" impedances and the "open-circuit transfer" impedance of a finite, purely reactive or dissipative, four-terminal network, are given (prescribed): to find the necessary and sufficient conditions to be imposed on the prescribed functions in order that they may be used as a family of admittance, or impedance functions as stated above, characterizing a passive structure.

This naturally also includes finding a general realization method by means of which at least one such structure can in every case be obtained.

SOME PREVIOUS CONTRIBUTIONS, ETC., RELATED TO OUR FIELD OF RESEARCH

The main contributions to network synthesis have dealt with one pair of terminals only, though extensions to purely reactive networks having two pairs of terminals, i.e. principally electric filter networks, and to four-terminal networks for a single frequency, also have been made. The extensions to filter network designs are principally built up on the theory underlying *Foster's* Reactance Theorem (20) published in 1924.

It is true that the fundamental properties of the electric filter network were discovered some years earlier by Wagner (14) and Campbell (15) and were further developed by Zobel,[18] but nevertheless Foster's paper highly stimulated the progress in filter design.

For many years the communication engineer, at least in U. S. A., has been using the Bell System method of design extensively, until recently *Cauer* (26) gave a more general theory which in practice may prove to be somewhat superior to the old method.[19]

The practical value of our contribution in this paper will not be discussed here, but unquestionably it will give us a most effective tool to handle the design of four-terminal electric communication networks, whether they are to be purely reactive or dissipative.

In his Reactance Theorem Foster gave the necessary and sufficient conditions, which must be satisfied by the impedance (or admittanee) function of a purely reactive two-terminal network, to be simply:

(i) *All zeros and poles must lie on the imaginary axis in the* λ*-plane.*

(ii) *The zeros and poles must mutually separate each other.*

[18] Bibliogr. 16–19.

[19] For a clear picture of Cauer's said method see *E. A. Guillemin* "A recent contribution to the design of electric filter networks." *Journal of Math. and Phys.*, June, 1922, Vol. XI, p. 150.

Due to the fact that complex or imaginary roots of an algebraic equation always occur in conjugate pairs, *the above two items include the necessity of either a pole or a zero at both the origin and at* ∞.

In the former case *the impedance-or admittance* function must consequently have the general form:

$$F_{(\lambda)} = \frac{a_0\lambda^{2n} + a_2\lambda^{2n-2} + a_4\lambda^{2n-4} + \ldots + a_{2n}}{b_1\lambda^{2n-1} + b_3\lambda^{2n-3} + \ldots\ldots + b_{2n-1}\lambda} \quad \ldots (2)$$

or the reciprocal of (2).

The above function can be directly expanded into *partial fractions*, and in this manner Foster (considering (2) as an imped-

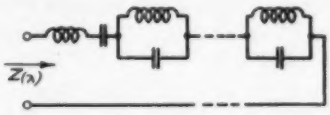


FIG. 7

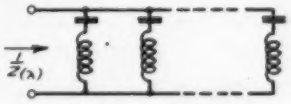


FIG. 8

ance function $Z_{(\lambda)}$) arrived at his structure shown in Fig. 7, consisting of a number of anti-resonance components plus an inductance (corresponding to the pole at ∞) and a condenser (corresponding to the pole at the origin) in series.

By inverting (2), and again expanding it into partial fractions, he arrived at a structure consisting of a number of resonance components in parallel as shown in Fig. 8.

On the other hand, we could as well have considered (2) as an admittance function $Y_{(\lambda)}$ in which case, after performing a partial fraction expansion, we would have obtained the structure of Fig. 9, or after having expanded its reciprocal into partial fractions we would get the structure of Fig. 10.

Cauer (22) proceeds further and shows that (when considering (2) as an impedance function $Z_{(\lambda)}$) by the use of a ***continued fraction*** expansion (finite Stieltjes (39) continued fraction) we can obtain two more networks equivalent to those in Figs. 7 and 8 respectively in form of the two ladder structures shown in Fig. 11.

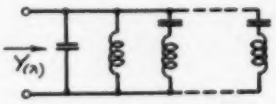


FIG. 9

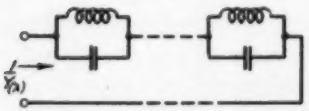


FIG. 10

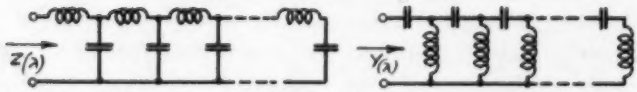


FIG. 11

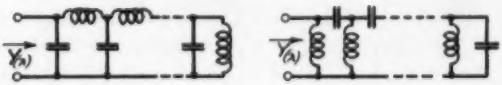


FIG. 12 FIG. 13

If he instead had considered (2) as an admittance function he would evidently, after a continued fraction expansion, have obtained the structure of Fig. 12, and then after a continued fraction expansion of (2), with its numerator and denominator divided by λ^{2n}, he would have obtained the structure of Fig. 13.[20]

Cauer (22) also solves the case when the prescribed impedance function corresponds to a *dissipative* two-terminal network *hav-*

[20] Note, a partial fraction expansion of (2) and its reciprocal gives different networks, while a continued fraction expansion gives the same network.

ing two kinds of elements only, i.e. resistance plus inductance or resistance plus capacitance. He shows that the necessary and sufficient conditions to be imposed on the impedance function corresponding to such two-terminal networks are simply:

(i) *All zeros and poles must lie on the negative real axis in the λ-plane.*
(ii) *Zeros and poles must mutually separate each other.*

It is interesting to know that (as shown in Part VIII) *from the sequence*[21] *of zeros and poles we may immediately, from the impedance- or admittance function, determine which one of the two combinations of elements the network is going to contain.*

The networks will be analogous to those shown in Figs. 7–13 when replacing inductances and condensers respectively with resistances in a proper way. Due to the fact, however, that, when considering the λ-plane sterographically projected upon a λ-sphere, the zeros and poles in this case do not lie on a closed "circle," as in the purely reactive case, but on a half circle, the separation requirement can consequently be fulfilled even if we have no zero or pole at the origin or at ∞.

In addition, the rule for the sequence of zeros and poles (note: the origin and ∞ are included therein) may in the dissipative case explicitly forbid a zero or a pole to lie at one of the two extreme points respectively. For instance, if the sequence begins with a pole (as being the case when $Z_{(\lambda)}$ corresponds to a network having resistance, plus capacitance elements only), then no zero can fall at the origin but possibly the initial pole may fall there, etc.

The continued fraction expansion method has later met with considerable interest from a number of investigators.

Fry (27) has discussed the possibility of applying a continued fraction expansion to any prescribed impedance function. An

[21] Bibliogr. 38, p. 3.

infinite continued fraction expansion can be used to give any desired degree of approximation, but Fry is, however, not concerned with an exact realization of his function, and does not reach a general solution of the problem.

This was, however, accomplished by *Brune* (38) who used a different method of attack, resulting in an *exact* realization of any prescribed impedance function corresponding to a finite, dissipative two-terminal network, whether it contains two or all three kinds of elements.

Thus *the necessary and sufficient conditions* to be satisfied by a function $f_{(\lambda)}$ in order to be the impedance function of a passive and finite two-terminal network Brune found to be:

(i) $f_{(\lambda)}$ *must be a rational fraction which is real for* λ *real.*
(ii) *The real part of* $f_{(\lambda)}$ *must be* $\geq$ *zero when the real part of* λ *is* $\geq$ *zero.*.

A function fulfilling the above requirements Brune calls a "*positive real*" function and gives several equivalent definitions of such a function.

Brune's whole method is based on a continuous removal of originally existing or produced poles on the boundary from his impedance function, and from its reciprocal, until finally a constant only is left.

When no more zeros or poles on the boundary are contained in the function as it stands, Brune produces a new zero there, generally at a finite frequency, by the removal of a positive constant, corresponding to a "minimum" resistance value, and producing a pole at infinity, making it possible to remove a conjugate pair of poles from the reciprocal of his function. Then after again having inverted the rest of the reciprocal of his function, i.e. being back to an impedance function, he can remove a new pole at ∞, and here one cycle is completed, decreasing the degree of λ in the numerator as well as in the denominator by two units.

The network arrived at in this manner will usually be as shown in Fig. 14 in which the two elements L_0 and C_0 may or may not be present.

In the evaluation procedure one of the inductances L_1, L_2 and L_3 comes out "negative," but their combination may, however, always be replaced by a pair of perfectly coupled "positive" inductances.

It is not always, however, that the network obtained by using Brune's method comes out as shown in Fig. 14. If in a cycle the lowest point of the Re $Z_{(j\omega)} - \omega$ curve happens to fall at ∞ (or at the origin respectively), then for this cycle the horizontal arm will contain resistance only and the admittance arm a condenser (an inductance respectively) only.

FIG. 14

Thus it is clear that in case the impedance function refers to a dissipative network having two kinds of elements only, in which case Re $Z_{(j\omega)} - \omega$ curve invariably has its lowest value at the origin or at ∞, then the corresponding network will be a ladder structure having condensers, or inductance coils respectively, in the admittance arms and resistances only in the horizontal branches.

Brune's method is probably the first general one known for the exact realization of any prescribed impedance function having a finite number of zeros and poles and corresponding to a dissipative two-terminal network. Like the networks found by partial fraction- or continued fraction expansion they contain *a minimum number of elements.*

This minimum number can directly be found to be equal to the number of independent arbitrary constants of the impedance function but the number of meshes in the network to be obtained

cannot beforehand be predicted because, as shown by *Howitt* (36), *every coefficient in the impedance function is actually a determinant of the numerical values of certain network elements*, and even if these elements are finite and positive the determinant itself may be zero.

It is obvious that two rational fractions, in order to be the impedance function of the same network, must be identical. The converse is not true, however, because *the same expression of the form* $Z_{(\lambda)} = \frac{f_{(\lambda)}}{h_{(\lambda)}}$ *may be the impedance function of any one of a number—perhaps an infinite set—of equivalent networks.*

For networks having all three kinds of elements, both Foster (21) and Cauer (22) make use of *equivalence equations*, but the procedure of finding equivalent networks by this method is very laborious.

By means of the equivalence equations and the mutual parameter plane, Howitt (36) arrived at an infinite set of equivalent networks having two kinds of elements.

More important, however, is his matrix method, which in certain cases is very simple and powerful for finding equivalent networks.

Every network determines one, two or three *quadratic forms* depending upon whether the network has one, two or three kinds of circuit elements, and these quadratic forms, which have their exact correspondence in dynamics and differential geometry, must all be *positive definite*.

Howitt showed, however, that we need to write down neither the Kirchhoff's differential equations for the network nor its quadratic forms, as both, so to say, vanish from the picture.

All we need to do is to perform a standard transformation on a matrix (one for each kind of elements present in the original network) which can be written down immediately by inspection only.

The method is equivalent to replacing the n independent, *instantaneous* mesh currents in Kirchhoff's equations for a given

network with another set of n mesh currents depending linearly upon the first set in such a manner as to leave the quadratic forms for the instantaneous rate of electromagnetic and electrostatic energy storage and heat invariant.

The above matrix method of finding equivalent networks has to be further developed, however, because it does not in every case give *all* equivalent networks, and further the coefficients in the linear affine transformation cannot be given arbitrary values if the equivalent network to be obtained is required to be *passive*.

It was actually found by the writer when working on the problem of *finding equivalent networks having a different number of meshes than the original one*, that the coefficients giving passive structures fall between certain very narrow limits.

So for instance in *the two-mesh case* they were found to fall within two very narrow ranges *along two straight lines*.

Thus there is another problem to be solved before Howitt's method acquires real practical importance,—and that is *to find a law for the coefficients in the affine transformation giving us the ranges within which, without use of ideal transformers, usable values of these coefficients can be picked out directly*.

The problem of finding equivalent networks having a different number of meshes (as mentioned above) is a very deep one and it is not yet fully understood. It seems evident that it cannot generally be done, but as pointed out by Brune and also found by the writer, it can at least be accomplished *in certain specific cases*.

In the foregoing we have mainly been dealing with two kinds of expansion, i.e. the partial fraction- and the continued fraction expansion. These are, however, only two methods of expressing a rational fraction, and several others may be used.

If a driving-point function of a two-terminal network is expanded into a convergent series the successive terms usually have different signs. Thus even if each term can be translated into a physically passive structure, we cannot handle the situation

because we only know how to add the separate structures, but not how to subtract them.

For a four-terminal network the case is different, because here we may work with the transfer function plus two image impedance functions. Here *a minus sign in front of a term in a series expansion of the transfer admittance (or impedance) simply means that one pair of terminals of the corresponding (component) network has to be reversed.*

This was the approach of *Lee* (37) in his doctor's dissertation. By using the Fourier Transforms of Laguerre's functions he obtained a series of orthogonal functions each corresponding to a lattice structure having certain properties, and by using an arbitrary number of terms he could get any degree of accuracy in approximating the transfer function.

It may be mentioned, however, that the Fourier- or Hilbert's transforms, used by Lee, for complicated functions become very laborious if not impossible to solve analytically.

It has long been known that the lattice structure has a large field of use in electrical engineering practice. The first one to show this fact was probably *Bartlett*,[22] who proved that when a symmetrical lattice structure is used as an equivalent representation of an artificial line, then in this lattice structure, when pictured as a bridge structure, two opposite impedances become each $= Z_0 \tan \mathrm{h} \frac{\theta l}{2}$, and the two others become each $= Z_0 \cot \mathrm{h} \frac{\theta l}{2}$, where Z_0 is the characteristic impedance of the line, θ is the propagation function, and l is the length of the line.

The interesting fact is that $Z_0 \tan \mathrm{h} \frac{\theta l}{2}$ and $Z_0 \cot \mathrm{h} \frac{\theta l}{2}$ can both be given definite physical interpretations,—being nothing but the "short-circuit driving-point"- and the "open-circuit driving-point" impedances respectively of a line of length $\frac{l}{2}$. Further, they can both be expanded into an infinite continued fraction, of

[22] See *Philosophical Magazine*, 1925, Vol. 49, p. 728.

which each term corresponds to a physically realizable impedance- or admittance-function.

If on the other hand the line were represented by a π or a T-structure we would have encountered impedances of the form $Z_0 \sin h\ \theta$ and $\frac{Z_0}{\sin h\ \theta}$ respectively, which cannot be given any physical interpretation in terms of the short-circuit- or open-circuit driving-point impedances, and further no expansion giving positive terms only is known for these two impedances.

Consequently, as an artificial line is a symmetrical network whose driving-point impedances or admittances at respective ends are equal, we have a right to assume that a set of rational, fractional functions, satisfying certain necessary and sufficient conditions to be given in the following, can always be given a physical representation in form of a symmetrical lattice-structure, if the two driving-point functions are equal.

This is actually the case, and it was later discussed by Cauer[23] in several articles.

With the material in Parts I to V of this paper available, it can also be proved, as shown under item (h) in Part VI. Note that this proof is given in terms of the members of one family only (the α:s).

In the same Part under items (a) and (b) an analogous proof is given for the fact that when the transfer function is equal to either one of the two driving-point functions, then our family of given functions can always be given a physical representation in form of an L-structure.

Also this fact has earlier been discussed by Cauer[24] and the practicability of using an L-structure in certain specific cases has previously been pointed out by Bartlett.[25]

A brief discussion of *Vaulot's*[26] work, which probably is the one closest related to our problem, will also be given.

[23] Bibliogr. 22, p. 356; Bibliogr. 23, p. 272. Bibliogr. 24, and Bibliogr. 26.
[24] Bibliogr. 23, p. 281.
[25] *Philosophical Magazine*, 1924, Vol. 48, p. 859.
[26] Bibliogr. 32 and 33. *Annales des Postes*, 1927, p. 860.

Vaulot, however, is dealing with a single frequency only, as explicitly stated in Bibliogr. 32, p. 502. This is also clear from his treatment as a whole when synthesizing a four-terminal network. He mainly deals with the four well known General Circuit "Constants" A B C D, which he splits up into their real and imaginary parts as:

$$\left.\begin{aligned} A &= a + j\,a' \\ B &= b + j\,b' \\ C &= c + j\,c' \\ D &= d + j\,d' \end{aligned}\right\} \qquad \ldots\ldots(3)$$

and considers each one of these parts as being constants and not functions of frequency.

The conditions Vaulot obtained and which he (at least originally) claims to be both necessary and sufficient are the following:[27]

$$bc + a'd' \geq 0 \qquad \ldots\ldots(4)$$
$$ac + a'c' \geq 0 \qquad \ldots\ldots(5)$$
$$cd + c'd' \geq 0 \qquad \ldots\ldots(6)$$
$$ab + a'b' \geq 0 \qquad \ldots\ldots(7)$$
$$bd + b'd' \geq 0 \qquad \ldots\ldots(8)$$

The physical meaning of above inequalities will be clear to us after having read Parts I to III. They will be discussed under item (h) of the last mentioned Part, and it can already here be pointed out, that due to some material to be found under items (a), (c), (e) and (g) of Part III, *three* of the above conditions can be dispensed with, and the only ones we need are (4) plus any one of the four others.

Later in his article Vaulot switches over from working with the four General Circuit "Constants" to three other constants, and it soon becomes clear that they correspond to our β:s at a single frequency.

[27] Note: In *Annales des Postes*, 1927, p. 860, Vaulot changes his attitude as to these conditions and says that they are necessary only,—not sufficient.

In this way, he arrives at five inequalities (in terms of the real and imaginary parts of his new constants) equivalent to (4)–(8) but also here, three of the conditions can immediately be dispensed with, because they depend upon the two remaining ones, of which, however, one must be the one corresponding to (4).

Lastly, it may be mentioned that in Bibliography 39–41 some research on synthesis of four-terminal networks can be found, although a general solution of the problem is not reached.

PART I

SOME DEVIATIONS AND FUNDAMENTAL RELATIONSHIPS, ETC.

(a). Suppose we have a linear, passive four-terminal network as shown in Fig. 15. Assuming the applied voltages at respec-

FIG. 15

tive ends to be sinusoidal of the form $Ee^{j\omega}$, then all mesh currents must necessarily be of the same form, i.e., $Ie^{j\omega}$ and the well known Kirchhoff's equations for said network are:

$$\left.\begin{aligned} z_{11}I_1 + z_{12}I_2 + z_{13}I_3 + \ldots\ldots\ldots z_{1n}I_n &= E_1 \\ z_{21}I_1 + z_{22}I_2 + z_{23}I_3 + \ldots\ldots\ldots z_{2n}I_n &= 0 \\ z_{31}I_1 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ z_{n1}I_1 + z_{n2}I_2 + z_{n3}I_3 + \ldots\ldots\ldots z_{nn}I_n &= -E_n \end{aligned}\right\} \ldots\ldots(9)$$

where $z_{ts} = z_{pt} = R_{st} + L_{st}\,p + \dfrac{S_{st}}{p}; \left(S_{st} = \dfrac{1}{C_{st}}\right)$; R, L, and C are the three kinds of network elements: resistance, inductance and capacitance.

Further $p = \dfrac{d}{dt}$

$\omega = 2\pi$ times frequency.

Letting D be the determinant of the z:s in (9) and M_{st} be the corresponding minor of the s^{th} row and the t^{th} column, then when

solving for the currents in the 1^{st} and n^{th} mesh, *being the only ones which interest us*, we get by the aid of Cramer's Rule:

$$\left.\begin{aligned} I_1 &= \frac{\begin{vmatrix} E_1 & z_{12} & z_{13} \dots & z_{1n} \\ 0 & \dots\dots & & z_{2n} \\ \dots & \dots\dots & & \dots \\ -E_n & z_{n2} & z_{n3} \dots & z_{nn} \end{vmatrix}}{D} = \frac{M_{11}}{D}E_1 + 0 + 0 + \dots - \frac{M_{n1}}{D}E_n \\ I_n &= \frac{\begin{vmatrix} z_{11} & z_{12} & z_{13} \dots & E_1 \\ z_{21} & z_{22} \dots & & 0 \\ \dots & \dots\dots & & \dots \\ z_{n1} & \dots\dots & & -E_n \end{vmatrix}}{D} = \frac{M_{1n}}{D}E_1 + 0 + 0 + \dots - \frac{M_{nn}}{D}E_n \end{aligned}\right\} \dots (10)$$

For short-circuit at the respective ends the corresponding voltages become zero, and when writing equations (10) as

$$\left.\begin{aligned} I_1 &= \alpha_{11}E_1 - \alpha_{1n}E_n \\ I_n &= \alpha_{1n}E_1 - \alpha_{nn}E_n \end{aligned}\right\} \quad \dots\dots (11)^{28}$$

and knowing the fact that *when dealing with quantities obtained at end n, either the sign of the two end currents or of the two end voltages has to be reversed*,[29] then we see that when $E_n = 0$:

$\alpha_{11} = \dfrac{M_{11}}{D} = \dfrac{I_1}{E_1}$, i.e. it is *the short-circuit driving-point admittance at end* 1; further when $E_1 = 0$:

$\alpha_{nn} = \dfrac{M_{nn}}{D} = \dfrac{I_n}{E_n}$, i.e. it is *the short-circuit driving-point admittance at end n*, and:

$\alpha_{1n} = \dfrac{M_{n1}}{D} = \dfrac{M_{1n}}{D} = \dfrac{I_1}{E_n} = \dfrac{I_n}{E_1}$ is evidently *the short-circuit transfer admittance between the two ends.*[30]

[28] Note, however, our α:s as derived above are essentially *operators* and ought to have been written $\alpha_{11(p)}$, etc.

[29] Let us call this law "*The sign rule.*"

[30] Note $\dfrac{M_{n1}}{D}$ is $= \dfrac{M_{1n}}{D}$ due to the *reciprocal theorem*; Bibliogr. 7, p. 33.

When solving for the two voltages contained in (11) we get:

$$\left.\begin{aligned} E_1 &= \frac{\alpha_{nn}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} I_1 - \frac{\alpha_{1n}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} I_n \\ E_n &= \frac{\alpha_{1n}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} I_1 - \frac{\alpha_{11}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} I_n \end{aligned}\right\} \quad \ldots\ldots(12)$$

For open-circuit at respective ends in Figure 15 the corresponding currents become zero, and when writing (12) as:

$$\left.\begin{aligned} E_1 &= \beta_{11} I_1 - \beta_{1n} I_n \\ E_n &= \beta_{1n} I_1 - \beta_{nn} I_n \end{aligned}\right\} \quad \ldots\ldots(13)^{31}$$

and applying the sign rule we see that:

$\beta_{11} = \dfrac{E_1}{I_1}$ is *the open-circuit driving-point impedance at end* 1.

$\beta_{nn} = \dfrac{E_n}{I_n}$ is *the open-circuit driving-point impedance at end n.*

$\beta_{1n} = \dfrac{E_1}{I_n} = \dfrac{E_n}{I_1}$ is *the open-circuit transfer impedance between the two ends.*

When again solving for the both currents contained in (13) we get:

$$\left.\begin{aligned} I_1 &= \frac{\beta_{nn}}{\beta_{11}\beta_{nn} - \beta_{1n}^2} E_1 - \frac{\beta_{1n}}{\beta_{11}\beta_{nn} - \beta_{1n}^2} E_n \\ I_n' &= \frac{\beta_{1n}}{\beta_{11}\beta_{nn} - \beta_{1n}^2} E_1 - \frac{\beta_{11}}{\beta_{11}\beta_{nn} - \beta_{1n}^2} E_n \end{aligned}\right\} \quad \ldots\ldots(14)$$

i.e. we are back to equations (11).

Consequently we have found that between our α:s and β:s the following relations hold:[32]

[31] Note, however, our β:s as derived above are essentially *operators* and ought to have been written $\beta_{11(p)}$, etc.

[32] Compare Bibliogr. 39.

$$\left.\begin{aligned} \alpha_{11} &= \frac{\beta_{nn}}{\beta_{11}\beta_{nn} - \beta_{1n}^2} \\ \alpha_{nn} &= \frac{\beta_{11}}{\beta_{11}\beta_{nn} - \beta_{1n}^2} \\ \alpha_{1n} &= \frac{\beta_{1n}}{\beta_{11}\beta_{nn} - \beta_{1n}^2} \end{aligned}\right\} \quad \ldots\ldots(15)^{33}$$

and:

$$\left.\begin{aligned} \beta_{11} &= \frac{\alpha_{nn}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} \\ \beta_{nn} &= \frac{\alpha_{11}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} \\ \beta_{1n} &= \frac{\alpha_{1n}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} \end{aligned}\right\} \quad \ldots\ldots(16)^{33}$$

Equations (15) and (16) reveal the interesting fact that generally *the α:s and β:s*, from a pure mathematical point of view, simply *are the elements of two non-singular inverse*[34] *square matrices*, the product of which is equal to the *unit matrix*, often called *idemfactor*. Thus the following matrix equation holds:

$$\begin{Vmatrix} \alpha_{11} & -\alpha_{1n} \\ \alpha_{1n} & -\alpha_{nn} \end{Vmatrix} \times \begin{Vmatrix} \beta_{11} & -\beta_{1n} \\ \beta_{1n} & -\beta_{nn} \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \quad \ldots\ldots(17)^{35}$$

[33] Note the reversal of subscripts on both sides of the equal sign in the two upper equations. This is necessary if we want our α:s and β:s both to have subscripts referring to corresponding end of the network. If for instance letting $\beta_{nn} = \frac{\alpha_{nn}}{\alpha\text{-determinant}}$ and $\beta_{11} = \frac{\alpha_{11}}{\alpha\text{-determinant}}$, then α_{11} and β_{nn} would refer to end 1, while α_{nn} and β_{11} would refer to end n, which would be confusing.

[34] A square matrix is said to be *non-singular* if its determinant is different from zero. If equal to zero the matrix is said to be *singular*. Bibliogr. 49, p. 65.

[35] Note the matrices we are dealing with are of the second order only. Further note that the cofactors and minors of these matrices are identical but for sign.

and consequently the following determinant equation also holds:

$$\begin{vmatrix} \alpha_{11} & -\alpha_{1n} \\ \alpha_{1n} & -\alpha_{nn} \end{vmatrix} \times \begin{vmatrix} \beta_{11} & -\beta_{1n} \\ \beta_{1n} & -\beta_{nn} \end{vmatrix} = +1 \qquad \ldots\ldots(18)$$

i.e. $$(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) = \frac{1}{\beta_{11}\beta_{nn} - \beta_{1n}^2} \qquad \ldots\ldots(18a)$$

From (18a) it is clear that *the zeros of the α-determinant are identical with the poles of the β-determinant and vice versa.*[36]

The prescribed matrix may be *singular*, however, in which case equations (15) and (16) do not apply, but also in this case the situation can readily be handled as will be shown later in Part XI.

The interpretation of our α:s and β:s is rather involved.

(1) Essentially they *are proportionality factors* between voltages and currents and vice versa.

(2) For an alternating impulse of unit strength they can be considered as *currents*[37] *and voltages* respectively.[37]

(3) When derived from Kirchhoff's equations as shown in Part I and replacing the ordinary ohmic resistance or conductance of these equations with operators, depending upon a known generalization of Ohm's law, they can be considered as *operators*[38] and can be given a physical interpretation only after they have operated on something.

(4) At the same time when writing down Kirchhoff's equations (one for each mesh of the network) and when then, by the aid of Cramer's Rule, eliminating all "internal" currents or voltages, which do not interest us, the differential equations characterizing the network at the four terminals remain, from which we see that our α:s and β:s, after having operated on a current or a voltage, are nothing but a *shorthand method of writing these remain-*

[36] For an illustration see *Example 1* (Appendix, p. 199).

[37] Note as the α:s and β:s are *not functions of time*, we must when considering them as currents and voltages respectively look upon them as vectors only.

[38] Essentially *"operator functions."*

ing differential equations (having constant coefficients) for the case when the current or voltage operated on is unity.

(**b**). Now let us express the four General Circuit "Constants" *ABCD* in terms of our α:s and β:s. These "Constants" will in our case be called the four *General Circuit Parameters*,[39] because they, when characterizing a four-terminal communication network, are not constants but *functions of frequency*.[39] They are defined by the equations.[40]

$$\left.\begin{aligned} E_1 &= A\,E_n + BI_n \\ I_1 &= C\,E_n + DI_n \end{aligned}\right\} \qquad \ldots\ldots(19)$$

On the other hand when solving for E_1 and I_1 from equations (11) we get:

$$\left.\begin{aligned} E_1 &= \frac{\alpha_{nn}}{\alpha_{1n}} E_n + \frac{1}{\alpha_{1n}} I_n \\ I_1 &= \frac{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2}{\alpha_{1n}} E_n + \frac{\alpha_{11}}{\alpha_{1n}} I_n \end{aligned}\right\} \qquad \ldots\ldots(20)$$

and when solving for the same voltage and current from equations (13) we get:

$$\left.\begin{aligned} E_1 &= \frac{\beta_{11}}{\beta_{1n}} E_n + \frac{\beta_{11}\beta_{nn} - \beta_{1n}^2}{\beta_{1n}} I_n \\ I_1 &= \frac{1}{\beta_{1n}} E_n + \frac{\beta_{nn}}{\beta_{1n}} I_n \end{aligned}\right\} \qquad \ldots\ldots(21)$$

[39] This name was suggested by Dr. *Vannevar Bush* to the writer who intended to use the somewhat inferior name "*General Circuit Functions*." The reason why ABCD ordinarily are called Constants is due to the fact that originally they have been used when handling power transmission line problems, in which case they actually are constants, because a single frequency only is considered. As in our case, they are functions of frequency, they ought to be written $A_{(\lambda)}$ $B_{(\lambda)}$ $C_{(\lambda)}$ and $D_{(\lambda)}$, but the subscript will be left out throughout.

[40] Bibliogr. 10, p. 246; Bibliogr. 25.

comparing equations (20) and (21) with (19) we have:

$$\left.\begin{aligned} A &= \frac{\alpha_{nn}}{\alpha_{1n}} = \frac{\beta_{11}}{\beta_{1n}} \\ B &= \frac{1}{\alpha_{1n}} = \frac{\beta_{11}\beta_{nn} - \beta_{1n}^2}{\beta_{1n}} \\ C &= \frac{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2}{\alpha_{1n}} = \frac{1}{\beta_{1n}} \\ D &= \frac{\alpha_{11}}{\alpha_{1n}} = \frac{\beta_{nn}}{\beta_{1n}} \end{aligned}\right\} \quad \ldots\ldots (22)$$

From (19) as well as (22) we see that the General Circuit Parameters *A and D have no dimension, while B has the dimension of an impedance C the dimension of an admittance.*

Further, from (22) it is clear that when α_{nn} is $= \alpha_{11}$ (requiring β_{11} to be $= \beta_{nn}$), then A becomes $= D$. Further, when α_{nn} is $= \alpha_{1n}$ (requiring β_{11} to be $= \beta_{1n}$) then A becomes $= 1$, and when α_{11} is $= \alpha_{1n}$ (requiring β_{nn} to be $= \beta_{1n}$) then D becomes $= 1$.

These facts are of great importance and they will be taken up in more detail in Part VI.

(c). In the foregoing we have stated that *three* functions completely specify a four-terminal network, but here we have *four* functions. Consequently, the four General Circuit Parameters must be tied together by a general relation between them.

This fact is generally known,[41] and it can be proved in a very simple manner by the aid of (22), as follows:

Using the quotients in α:s (say) of these equations and forming the difference $(AD - BC)$ we get:

$$\frac{\alpha_{nn}}{\alpha_{1n}}\frac{\alpha_{11}}{\alpha_{1n}} - \frac{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2}{\alpha_{1n}\alpha_{1n}}$$

which expression reduces to $+1$.

[41] See Bibliogr. 34, p. 939; *Rosebrugh, T. R., Trans. A. I. E. E.*, 1930, Vol. 49, p. 1419.

Thus, the general relation is simply:

$$(AD - BC) = 1 \qquad \ldots\ldots(23)$$

It may be pointed out, however, that the relation between the four General Circuit Parameters depends upon what sign we give E_n in equations (9). If instead of writing $-E_n$ we had written $+E_n$, then the last term of equations (11) would have had a plus sign instead of a minus sign, making the two terms $\frac{\alpha_{nn}}{\alpha_{1n}}$ and $\frac{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2}{\alpha_{1n}}$ of equations (20) have minus signs instead of plus sign, i.e. the difference $AD - BC$, would have become equal to -1.

This would influence another transformation in a very interesting manner.

When solving for E_n and I_n in (19) *by merely substitution* we get the generally known equations:

$$\left.\begin{aligned} E_n &= DE_1 - BI_1 \\ I_n &= -CE_1 + AI_1 \end{aligned}\right\} \qquad \ldots\ldots(24)$$

When solving for the same quantities in (19) *by the aid of Cramer's Rule* we get, however:

$$\left.\begin{aligned} E_n &= \frac{\begin{vmatrix} E_1 & B \\ I_1 & D \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} = \frac{\begin{vmatrix} E_1 & B \\ I_1 & D \end{vmatrix}}{\Delta} = \frac{D}{\Delta}E_1(-1)^2 + \frac{B}{\Delta}I_1(-1)^3 = \frac{D}{\Delta}E_1 - \frac{B}{\Delta}I_1 \\ I_n &= \frac{\begin{vmatrix} A & E_1 \\ C & I_1 \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} = \frac{\begin{vmatrix} A & E_1 \\ C & I_1 \end{vmatrix}}{\Delta} = \frac{C}{\Delta}E_1(-1)^3 + \frac{A}{\Delta}I_1(-1)^4 = -\frac{C}{\Delta}E_1 + \frac{A}{\Delta}I_1 \end{aligned}\right\} \ldots(25)$$

Thus equations (25) will be identical with equations (24) only if Δ is $= +1$.

(d). Besides (23) another interesting relation is hidden in equations (22).

Forming the product AD in two different ways we get:

$$\frac{\alpha_{nn}\beta_{nn}}{\alpha_{1n}\beta_{1n}} = \frac{\alpha_{11}\beta_{11}}{\alpha_{1n}\beta_{1n}}$$

i.e.

$$\alpha_{11}\beta_{11} = \alpha_{nn}\beta_{nn} \qquad \ldots\ldots(26)$$

Equation (26) tells us that: *the products of the "short-circuit driving-point" admittance and the "open-circuit driving-point" impedance at respective ends of a four-terminal network are equal.*

PART II

THE "REALITY" REQUIREMENT, LOCATION OF ZEROS AND POLES, AND SIGN OF COEFFICIENTS, ETC.

Since each family of our functions characterize a four-terminal network having a finite number of meshes and since each separate function is a rational function, it is clear that we must be able to write every α and β as a rational fraction. This is indicated by equations (1) for the α:s and when comparing (16) and (18a) with (1) we see that the β:s as well as the two determinants, fulfil the same requirement.

(**a**). Each one of the α:s and β:s must further *be real for* λ *real*, which is the same as saying that for $\omega = 0$ the response vector must be in phase with the impulse vector and both vectors must have the same direction, this being obvious because we actually are dealing with the D. C. case.

(**b**). As to the *location of zeros and poles* of *the driving-point functions* it is known[42] that they all must lie in the left half of the complex λ-plane including the boundary.

This is clear from the fact that the above "functions," as derived from equations (9) essentially are operators, and when operating on something (a voltage or a current) the expressions can be considered as the nonhomogenous linear differential equations, with constant coefficients, which remain after all "internal" currents or voltages respectively have been eliminated.

The solution of such an equation is always the sum of a "particular integral" (corresponding to the steady state term in the language of electrical engineers) and a "complementary function" (corresponding to the transient term).

[42] Bibliogr. 38, pp. 19–20.

The transient term can be said to be "independent" (except for certain arbitrary constants) of the impressed voltage or current, being a solution of any one of the equations:

$$\left.\begin{aligned} \alpha_{11}E &= 0 \\ \alpha_{nn}E &= 0 \\ \beta_{11}I &= 0 \\ \beta_{nn}I &= 0 \end{aligned}\right\} \qquad \ldots\ldots(27)$$

Letting the *zeros* of any one of the above functions be:

$$\lambda_1 = \gamma_1 + j\omega_1;\ \lambda_2 = \gamma_2 + j\omega_2;\ \lambda_3 = \gamma_3 + j\omega_3;\ etc.$$

then the transient voltage obtained from either one of the first two of equations (27) will be of the form:

$$E_{tr} = A_1\epsilon^{\lambda_1 t} + A_2\epsilon^{\lambda_2 t} + A_3\epsilon^{\lambda_3 t} + \ldots\ldots\ldots =$$

$$A_1\epsilon^{\gamma_1 t}(\cos\,\omega_1 t + j\,\sin\,\omega_1 t) + A_2\epsilon^{\gamma_2 t}\,(\cos\,\omega_2 t +$$

$$j\,\sin\,\omega_2 t) + \ldots\ldots\ldots \qquad \ldots\ldots(28)$$

The transient currents obtained from either one of the last two of equations (27) has the same form as (28), i.e.

$$I_{tr} = B_1\epsilon^{\gamma_1 t}(\cos\,\omega_1 t + j\,\sin\,\omega_1 t) + B_2\epsilon^{\gamma_2 t}(\cos\,\omega_2 t +$$

$$j\,\sin\,\omega_2 t) + \ldots\ldots\ldots \qquad \ldots\ldots(29)$$

From (28) and (29) we see, that all γ:s must each be ≤ 0, as otherwise the transient voltage or transient current, instead of ultimately vanishing to zero as required by the *Law of Conservation of Energy*, would exponentially become infinite with time, if the real part of one single zero of equations (27) should happen to be positive.

When reaching this conclusion, we have not mentioned the possibility of a term, having a positive γ in its exponent, becoming zero due to the fact that the integration constant, being the coefficient of the term in question, may happen to be zero.

The integration constants, however, are determined by the initial conditions, and as the network must be passive for any set of such conditions, the transient voltages or currents must die out for any value of the constants, and these values are determined according to a certain set of conditions.

Thus if a certain constant comes out $= 0$ in one case, it is $\neq 0$ in another.

Consequently all zeros of our four driving-point functions must lie in the left half of the λ*-plane including the boundary.*

The location of the *poles* can be handled in a similar manner by working with the reciprocal of the driving-point α:s and β:s in equations (27), i.e. when writing:

$$\left.\begin{aligned} \frac{1}{\alpha_{11}} I' &= 0 \\ \frac{1}{\alpha_{nn}} I' &= 0 \\ \frac{1}{\beta_{11}} E' &= 0 \\ \frac{1}{\beta_{nn}} E' &= 0 \end{aligned}\right\} \qquad \ldots\ldots (30)$$

then we get transient currents and transient voltages respectively of exactly the same form as those shown in equations (29) and (28), and applying the same reasoning as above we reach the requirement: *All poles of our four driving-point functions must lie in the left half of the* λ*-plane including the boundary.*

A pole to the right of the boundary would imply that once an oscillation at a certain frequency were established it would persist forever with constantly increasing amplitude, even after the force has been removed.

The location of the zeros and poles of our driving-point functions could, however, as well have been determined by the aid

of Heaviside's expansion formula[43] for a constant "unit" voltage and "unit" direct current respectively applied.

(c). As to the *transfer function* (α_{1n} or β_{1n} respectively), its *poles* are generally the same as those of the two driving-point functions, and consequently *they all lie in the left half of the* λ-*plane including the boundary.*

The *zeros* of the transfer function, however, are found to make an exception as to the restriction of their location! They may lie at any[44] point in the λ-plane.

This may not seem to be clear at the first sight and it may even seem to be a contradiction to the Law of Conservation of Energy, because suppose our network is shorted at end n.

Then equations (11) go over into:

$$\left.\begin{aligned} I_1 &= \alpha_{11}E_1 \\ I_n &= \alpha_{1n}E_1 \end{aligned}\right\} \qquad \ldots\ldots(31)$$

We know that as to the first equation of (31) we may solve for the transient voltage for a current applied or for a transient current for a voltage applied giving us the first equation in (27) and (30) with solutions as shown in (28) and (29) respectively.

If now solving for the transient voltage from the equation:

$$\alpha_{1n}E_1 = 0 \qquad \ldots\ldots(32)$$

giving us a "solution" of the form (28), and knowing that in this case any one of the γ:s very well can be positive, it seems as if the transient voltage at end 1 might become infinitely large.

We have here started out from an impossible assumption, however, *because for* $E_n = 0$ *we cannot assume a current* I_n *applied.*

The only possibility is having an I_n *flowing for a voltage* E_1

[43] Bibliogr. 4; Bibliogr. 5; Bibliogr. 38, p. 19.

[44] Why this is allowed will be clear in the following after having read this and the next Part, and after having been acquainted with Cauchy's and Poisson's integral equations. It is demonstrated by *Example* 2 (Appendix, p. 200) which alone is a sufficient proof for the above mentioned fact, because the transfer function is there derived from a passive four-terminal network.

applied, i.e. instead of solving for the transient voltage from (32) we will have to solve for the transient current from:

$$\frac{I_n}{\alpha_{1n}} = 0 \qquad \ldots\ldots(33)$$

i.e. not the numerator of α_{1n} but the denominator is coming into play.

In an analogous manner for open-circuit at end n we get from (13)

$$\left.\begin{aligned} E_1 &= \beta_{11} I_1 \\ E_n &= \beta_{1n} I_1 \end{aligned}\right\} \qquad \ldots\ldots(34)$$

To solve for the transient current from:

$$\beta_{1n} I_1 = 0 \qquad \ldots\ldots(35)$$

would be absurd, but the equation:

$$\frac{E_n}{\beta_{1n}} = 0 \qquad \ldots\ldots(36)$$

which is physically possible, gives us a perfectly correct result as to the transient voltage at end n for a current applied at end 1 in agreement with the Conservation of Energy Law.

Thus we see that the location of both zeros and poles of the driving-point functions is guided by the above Energy Law, but as to the transfer function poles only are required to lie in the left half of the λ-plane, *while zeros of this function may lie anywhere in the plane without violating the law.*

(d). As to the *location of zeros and poles of the α- and β-determinants*, i.e. of $(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2)$ and $(\beta_{11}\beta_{nn} - \beta_{1n}^2)$ respectively, it is immediately clear from equations (15), (16) and (18a) that *they all must lie in the left half of the complex plane including the boundary.*

This fact can also readily be proved when considering the Law

of Conservation of Energy and in a manner similar to the one shown under item (b).

(e). Regarding the *sign* of the coefficients in the numerator and denominator of each separate function and of the two determinants, it is known that if all roots of an integral polynomial in λ are lying in the left half of the plane including the boundary, then this fact is necessary and sufficient for requiring every term of the polynomial to have a positive sign. If the roots are lying at any place in the plane, then the sign of the coefficients may be positive or negative, or some positive while others are negative.

Thus *every term in the numerator as well as in the denominator of our driving-point functions and the two determinants and further of the denominator of the transfer function of each family must necessarily be positive, while each separate term in the numerator of the transfer function may have any sign,*[45] *and for this function a change of sign for all terms in the numerator simply corresponds to a reversal of a pair of terminals.*

(f). When going in the opposite direction, i.e. from sign to location, a positive sign of the coefficients is only a necessary condition for the roots to lie in the left half of the λ-plane, but not sufficient.

Besides having positive sign the coefficients must then satisfy the Hurwitz (50) conditions[46] imposed on the determinants and principal minors of said coefficients.

In case the power of λ in a function is very high, it would be laborious to find out where zeros and poles lie due to the difficulty of solving high degree equations whose roots generally are complex.

The only integral polynomials which interest us here, however, are those whose roots have restricted location, and *thus we*

[45] For an illustration see *Example* 2 (Appendix, p. 200).
[46] Bibliogr. 50 and 51; Bibliogr. 38, p. 23.

understand, what simplification in handling the matter the use of the Hurwitz conditions offer.

(g). For the case when the power of λ in our α:s or β:s is high, it is shown in the preceding item, that we have a possibility to check up whether zeros and poles lie in the left half of the λ-plane by the aid of the Hurwitz conditions. This is very important, because solving the integral polynomials in λ (equated to zero) forming the numerator or the denominator of an α- or a β-function may, in some instances, be practically impossible.

Besides λ-functions *we are also dealing with ω-functions, however, and these may happen to be of a very high degree.* As also in this case the roots may be real or complex, it might be practically impossible to obtain them by a straight forward procedure.

Therefore here we also need some auxiliary method, and it is readily found to be embodied in Sturm's well known Theorem,[47] which states that if we count the number of variations in sign of a number of functions for a definite value of the variable, and if for another value of the variable we get another number of variations, then we know that the number of real roots lying between the two values substituted is equal to the difference in the number of variations in sign.

Therefore the Theorem is very useful for locating real roots by making the range within which a certain root lies more and more narrow, *but in our case we are going to use it in a different manner.*

The functions we are going to apply the Theorem to (when necessary) are the real parts of the two driving-point functions and the determinant of the real parts.

In the next Part it will be shown that all these expressions have to be $\geq$ zero for all ω^2:s between[48] 0 and $+\infty$, i.e. no real roots are allowed to fall in between these two extreme ω^2 values.

[47] Bibliogr. 46, p. 472.

[48] Naturally this is the same as to say that the expressions have to be $\geq$ zero for all real ω:s.

Consequently when substituting $\omega^2 = 0$ and $\omega^2 = +\infty$ respectively[49] in Sturm's functions, the number of sign variations should be the same.

Further, as the sign of the terms containing ω^2 to the highest (as well as lowest) power always is positive, we then know that *if the same number of sign variations are obtained, for* $\omega^2 = 0$ *and* $\omega^2 = +\infty$ *then the curve stays above the* ω^2*-axis for all positive* ω^2*-values.*

In case the curve touches the axis, i.e. it has a double zero there, then the last Sturm-function is zero.

For an illustration of the use of the Theorem see *Example 3.*[50]

[49] Note the simplicity of these two substitutions.
[50] Appendix, p. 201.

Part III

SOME POSITIVENESS- AND NON-POSITIVENESS REQUIREMENTS AND RELATIONSHIPS, ETC.

We are now going to discuss the positiveness- and non-positiveness requirement for the real part of each driving-point function and of the transfer function and derive an important and fundamental determinant condition, always to be satisfied.

Further, it shall be shown that necessary conditions need to be imposed upon the α:s only, because then the β:s also satisfy the same conditions, and vice versa.

(**a**). As to the driving-point functions it is known that the real part of each one must be ≥ 0 over the whole right half of the λ-plane.

Considering the β:s, say, then the conditions $Re\beta_{11(j\omega)}$[51] ≥ 0 and $Re\beta_{nn(j\omega)} \geq 0$ can readily be proved to be necessary by considering an applied voltage of the form:[52]

$$E = E_m \cos \omega t = E_m Re\epsilon^{j\omega t} \qquad \ldots\ldots(37)$$

(where E_m is the amplitude value) causing a steady state current:

$$I_{st} = E_m Re\frac{\epsilon^{j\omega t}}{\beta_{(j\omega)}} = \frac{E_m}{|\beta_{(j\omega)}|}\cos(\omega t - \varphi) \qquad \ldots\ldots(38)$$

to flow; (note $\varphi = arg\beta_{(j\omega)}$).

After a sufficient long time the transient current (I_{tr}) in a dissipative network is negligible in comparison with I_{st}, and the consumed power (P) must be positive.

[51] The subscript $(j\omega)$ always indicates that we are considering the boundary only.

[52] Bibliogr. 38, p. 20.

Thus, over a period we get:

$$P = \frac{1}{T}\int_0^T EI_{st}\,dt = \frac{1}{T}\int_0^T E_m \cos \omega t \frac{E_m}{|\beta_{(j\omega)}|}\cos(\omega t - \varphi)\,dt =$$

$$= \frac{E_m^2}{T|\beta_{(j\omega)}|}\int_0^T \cos \omega t \cos(\omega t - \varphi)\,dt = \frac{E_m^2}{2|\beta_{(j\omega)}|}\cos \varphi \quad \ldots(39)$$

From (39) we see that P stays $\geq$ zero for φ between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, which is the same as to say that $Re\beta_{11(j\omega)}$ and $Re\beta_{nn(j\omega)}$ must each be $\geq$ zero.

Then, from the following under this Part we also know that $Re\alpha_{11(j\omega)}$ and $Re\alpha_{nn(j\omega)}$ *are* each ≥ 0, which can be proved to be necessary by considering an applied *current* of the form:

$$I = I_m Re\epsilon^{j\omega t} \quad \ldots\ldots(40)$$

causing a steady state voltage:

$$E_{st} = I_m Re\frac{\epsilon^{j\omega t}}{\alpha_{(j\omega)}} = \frac{I_m}{|\alpha_{(j\omega)}|}\cos(\omega t - \varphi) \quad \ldots\ldots(41)$$

to build up. Thereafter the procedure is exactly the same as demonstrated when considering the real parts of β_{11} and β_{nn}.

Consequently, we have obtained the necessary conditions (valid for all real values of ω):

$$\left.\begin{aligned} Re\ \alpha_{11(j\omega)} &\geq 0 \\ Re\ \alpha_{nn(j\omega)} &\geq 0 \\ Re\ \beta_{11(j\omega)} &\geq 0 \\ Re\text{-}\beta_{nn(j\omega)} &\geq 0 \end{aligned}\right\} \quad \ldots\ldots(42)$$

When in the following the subscript $(j\omega)$ is omitted, then our potential functions, instead of representing *curves* above and

along the boundary, represent *sheets* above the whole right half of the λ-plane.[53]

(**b**). *The real part of the transfer function* is not, however, limited by the same restriction as the real part of the driving-point functions.

This is clear from the physical fact that *the transfer function cannot be given any physical interpretation in terms of energy flow*, α_{1n} being a proportionality factor between output current and input voltage, and the reverse being true as to β_{1n}.

Thus, it has been found that *the real part of the transfer function may be positive or negative,—or positive for some ω:s while negative for others.*[54] A sufficient proof for this fact is given by a numerical example.[55]

(**c**). Below *we shall now derive a necessary condition which has proven to be of fundamental importance* for a successful treatment of our whole problem.

Let us terminate our four-terminal network in a passive two-terminal network at end n as shown in Fig. 16.

Then, by the aid of equations (19) we get:[56]

$$Z' = \frac{E_1}{I_1} = \frac{AE_n + BI_n}{CE_n + DI_n} = \frac{AZ + B}{CZ + D} \quad \ldots\ldots (43)$$

From (43) it is evident that: Z' is the driving-point impedance looking in at end 1 for any termination at the opposite end,

[53] Then we must add, however: "for $\mathrm{Re}\lambda \geq 0$."
Note: equations (42) were actually obtained already under Introduction on page 9 when discussing the direction of the response vector for a response at the end where the impulse is applied.

[54] Note, however, *the absolute value of the real part of the transfer function is not arbitrary*, as will be shown under the next item of this Part.

[55] See *Example 2* (Appendix p. 200) from which it is clear that $\mathrm{Re}\,\alpha_{1n(j\omega)}$ is > 0 for $\omega:\mathrm{s} < 1$, $= 0$ for $\omega = 1$, and < 0 for $\omega:\mathrm{s} > 1$. Note further that α_{1n} has a zero in the right half of the λ-plane, which simply forces $\mathrm{Re}\,\alpha_{1n(j\omega)}$ to be < 0 at some point on the boundary due to Poisson's integral requirement to be dealt with in the next Part.

[56] Note Z' and Z are both functions of λ and ought to have been written $Z'_{(\lambda)}$ and $Z_{(\lambda)}$ respectively, but for the sake of simplicity the subscript (λ) will be left out throughout.

including short- and open-circuit. Z is the driving-point impedance of the two-terminal network used for termination.

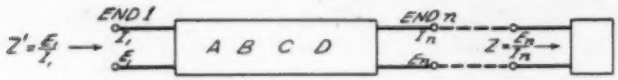


FIG. 16

When substituting for the General Circuit Parameters their relations *in our α:s (say)*, we have:

$$Z' = \frac{\alpha_{nn} Z + 1}{(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) Z + \alpha_{11}} \qquad \ldots\ldots(44)$$

Notations: At this point let us adopt the following notations when splitting up the functions (note all are generally complex) of (43) into their real and imaginary part:

$$Z' = Z'_r + jZ'_x \qquad \ldots\ldots(45)^{87}$$

$$Z = Z_r + jZ_x \qquad \ldots\ldots(46)^{87}$$

further:

$$\left.\begin{aligned} \alpha_{11} &= r_1 + jx_1 \\ \alpha_{nn} &= r_n + jx_n \\ \alpha_{1n} &= r_{1n} + jx_{1n} \end{aligned}\right\} \qquad \ldots\ldots(47)^{87}$$

Substituting (45), (46) and (47) into (44), we get:

$$(Z'_r + jZ'_x) = \frac{(r_n + jx_n)(Z_r + jZ_x) + 1}{\begin{aligned}&[(r_1 r_n - r_{1n}^2) - (x_1 x_n - x_{1n}^2) + j(r_1 x_n + \\ &+ r_n x_1 - 2r_{1n}x_{1n})](Z_r + jZ_x) + (r_1 + jx_1)\end{aligned}} \qquad \ldots\ldots(48)$$

[87] Note, all these real and imaginary parts are here referring to the whole right half of the λ-plane, i.e., each is a function of both γ and ω and ought to have an additional subscript $(\gamma\omega)$. This subscript, however, will be left out. When referring to the boundary only, they will, for the sake of simplicity, have the same appearance, but a subscript $(j\omega)$ will be added to the functions at the left side of the equal sign. The same holds for our β:s when split up in a similar manner, to be shown in the following, or other functions to be dealt with. If it should not seem clear, as to the number of variables it will be explicitly stated under the corresponding item.

In the continuation of this Part we are dealing with the whole right half of the λ-plane.

Then, when solving for the real part (which is the only one which interests us) we obtain:

$$Z_r' = \frac{\begin{array}{l}(Z_r^2 + Z_x)\,[(r_1r_n - r_{1n}^2)r_n + r_nx_{1n}^2 + r_1x_n^2 - \\ - 2x_nr_{1n}x_{1n}] + Z_r(2r_1r_n - r_{1n}^2 + x_{1n}^2) + \\ + Z_x(2r_{1n}x_{1n} - 2r_1x_n) + r_1\end{array}}{\begin{array}{l}[Z_r(r_1r_n - r_{1n}^2 - x_1x_n + x_{1n}^2) - Z_x(r_1x_n + r_nx_1 - \\ - 2r_{1n}x_{1n}) + r_1]^2 + [Z_r(r_1x_n + r_nx_1 - 2r_{1n}x_{1n}) + \\ + Z_x(r_1r_n - r_{1n}^2 - x_1x_n + x_{1n}^2) + x_1]^2\end{array}} \quad \ldots(49)^{58}$$

If instead of having terminated at end n, as shown in Fig. 16, we had terminated at end 1 in a passive two-terminal network whose impedance is $Z = Z_r + jZ_x$, then by the aid of equations (24), our "sign rule" and equations (46) and (47) we would have obtained the real part Z_r'' (say) of the driving-point impedance looking in at end n to be:

$$Z_r'' = \frac{\begin{array}{l}(Z_r^2 + Z_x^2)\,[(r_1r_n - r_{1n}^2)r_1 + r_1x_{1n}^2 + r_nx_1^2 - \\ - 2x_1r_{1n}x_{1n}] + Z_r(2r_1r_n - r_{1n}^2 + x_{1n}^2) + \\ + Z_x(2r_{1n}x_{1n} - 2r_nx_1) + r_n\end{array}}{\begin{array}{l}[Z_r(r_1r_n - r_{1n}^2 - x_1x_n + x_{1n}^2) - Z_x(r_nx_1 + r_1x_n - \\ - 2r_{1n}x_{1n}) + r_n]^2 + [Z_r(r_nx_1 + r_1x_n - 2r_{1n}x_{1n}) + \\ + Z_x(r_1r_n - r_{1n}^2 - x_1x_n + x_{1n}^2) + x_n]^2\end{array}} \quad \ldots\ldots(50)$$

Note, (49) and (50) are essentially the same equations, except the subscript for the real and imaginary parts of α_{11} and α_{nn} are reversed, which could have been expected. In the following let us deal with (49) only.

Let us impose Brune's conditions as stated under "introduction" upon Z' in (44) and Z_r' in (49) respectively to hold for any value and character of the passive termination, i.e. of Z in(46).

It is immediately clear that (44) fulfils item (i) of said conditions.

Then, (49) has to be *made* to fulfil item (ii), i.e. conditions have

[58] Note, the lower lines are a continuation of the upper line (holds for numerator as well as denominator).

to be imposed upon the r:s (and perhaps upon the x:s) as to keep every point of the Z_r'-sheet located above the λ-plane for all real ω:s and all real γ:s ≥ 0 and for any passive termination, i.e. corresponding to any positive real value of $Z_r \geq 0$ and any real value of Z_x.

The denominator of (49) is always positive, being the sum of two squares, and thus we need to consider the numerator (which has the dimensions of an admittance) only.

Denoting it by G we get:

$$\begin{aligned} G = {} & (Z_r^2 + Z_x^2)\,[(r_1 r_n - r_{1n}^2)\,r_n + r_n x_{1n}^2 + r_1 x_n^2 - \\ & - 2x_n r_{1n} x_{1n}] + Z_r(2r_1 r_n - r_{1n}^2 + x_{1n}^2) + \\ & + Z_x(2r_{1n}x_{1n} - 2r_1 x_n) + r_1 \qquad \ldots\ldots(51) \end{aligned}$$

Letting:[59]

$$\left.\begin{aligned} (r_1 r_n - r_{1n}^2)\,r_n + r_n x_{1n}^2 + r_1 x_n^2 - 2x_n r_{1n} x_{1n} &= a \\ (2r_1 r_n - r_{1n}^2 + x_{1n}^2) &= b \\ (2r_{1n}x_{1n} - 2r_1 x_n) &= c \\ r_1 &= d \end{aligned}\right\} \ldots (52)$$

Then (51) goes over into:

$$G = Z_r^2\,a + Z_x^2\,a + Z_r b + Z_x c + d \qquad \ldots\ldots(53)$$

For G equal to a constant (53) is evidently, for a certain γ and ω, representing a circle in a plane located above the Z_r- Z_x-plane, at a distance equal to this constant G-value. When letting γ and ω vary, at the same time as G is allowed to take on any value ≥ 0, for $Z_r \geq 0$, it is clear that (53) *is the equation of a moving paraboloid of revolution having a variable parameter*[60] $\left(= \frac{1}{a}\right)$.

[59] Note "a," "b," "c," and "d" are all functions of both γ and ω. Further, it is clear that "a" has the dimension of an admittance to the third power, "b" and "c" each of an admittance to the second power, and "d" of an admittance to the first power.

[60] Lotus rectum.

This fact can most readily be seen when rewriting (53) as:

$$G = a\left(Z_r + \frac{b}{2a}\right)^2 + a\left(Z_x + \frac{c}{2a}\right)^2 \left(\frac{4ad - b^2 - c^2}{4a}\right) \ldots (54)$$

The corresponding paraboloid is shown in Fig. 17. Note, however, the important fact that Z_r *in* (54) *is not allowed to take on negative values.*

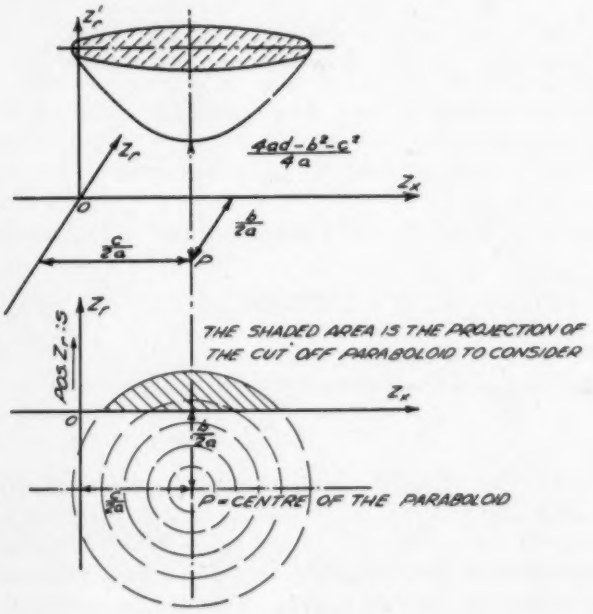


FIG. 17

This simply means that we cannot deal with the whole paraboloid but only with that part of it whose horizontal projection falls in that half of the Z_r-Z_x-plane where we have positive (or zero) Z_r-values, as indicated with the shaded area in the lower projection of Fig. 17.

Provided "a" is ≥ 0 for $\gamma \geq 0$, it is immediately clear from the form of (54) *that if the G-sheet*[61] *is* ≥ 0 *for* $Z_r = 0$, *then it certainly is* ≥ 0 *for a finite positive* Z_r value.

As to the positiveness of "a," let us see what the letter stands for physically:

Letting Z_r and Z_x increase towards infinity (corresponding to open circuit at end n), then (49) goes over into:

$$Z'_r = \frac{(r_1 r_n - r_{1n}^2) r_n + r_n x_{1n}^2 + r_1 x_n - 2 x_n r_{1n} x_{1n}}{(r_1 r_n - r_{1n}^2 - x_1 x_n + x_{1n}^2)^2 + (r_1 x_n + r_n x_1 - 2 r_{1n} x_{1n})^2} \quad \ldots\ldots(55)$$

which simply is the real part of β_{11}, as can be seen from (16).

The denominator is a sum of two squares, i.e. always positive, and consequently because $Re\beta_{11}$ is ≥ 0 for $\gamma \geq 0$ we see that "a," which simply is identical with the numerator of (55), *is* ≥ 0 for $\gamma \geq 0$.

For $Z_r = 0$ and any real value of Z_x equation (53) goes over into:

$$G = Z_x^2 a + Z_x c + d \quad \ldots\ldots(56)$$

which can be rewritten as:

$$G = a\left(Z_x + \frac{c}{2a}\right)^2 + \frac{4ad - c^2}{4a} \quad \ldots\ldots(57)$$

This is evidently the equation of a moving parabola, with variable parameter, cut out from the paraboloid in Fig. 17 by the G-Z_x*-plane.*

The parabola is shown in Fig. 18. *The vertex of this parabola must always stay above the* Z_x*-axis.* Thus the expression:

$$\frac{4ad - c^2}{4a}$$

must always be ≥ 0, which is equivalent to saying that *the discriminant of* (56), i.e. $(c^2 - 4ad)$, *must always be* ≤ 0.

[61] Above the right half of the λ-plane.

When substituting into the above discriminant the functions for a, c, and d, in equations (52), we get the inequality:

$$\left.\begin{aligned}(2\,r_{1n}x_{1n} - 2\,r_1x_n)^2 - 4\,[(r_1r_n - r_{1n}^2)\,r_n + \\ + r_nx_{1n}^2 + r_1x_n^2 - 2\,x_nr_{1n}x_{1n}]\,r_1 \leq 0\end{aligned}\right\} \ldots\ldots(58)$$

which reduces to:

$$-4(r_1r_n - r_{1n}^2)\,(r_1r_n + x_{1n}^2) \leq 0 \qquad \ldots\ldots(59)$$

The factor to the right is always positive because r_1 and r_n are each always ≥ 0, and *consequently the only requirement for*

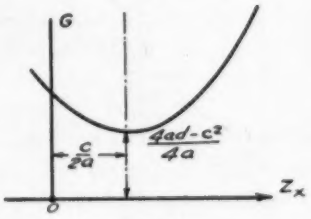


FIG. 18

(59) to always be ≤ 0 *is that the r-determinant is kept* ≥ 0 *for every* $\gamma \geq 0$ *and any real* ω.

Thus, the inequality:

$$(r_1r_n - r_{1n}^2) \geq 0, \textit{ for } \gamma \geq 0 \qquad \ldots\ldots(60)$$

(supplemented with either one of the conditions $r_1 \geq 0$ for $\gamma \geq 0$ *or* $r_n \geq 0$ for $\gamma \geq 0$) *is a necessary condition*[62] *which always must be satisfied!*

This fixes the sign of "b" in (52) *to be positive,* and as a result we see, from Fig. 17, that *the axis of the moving paraboloid never crosses the* Z_x*-axis* (because now we know that the ratio $\frac{b}{2a}$ always is ≥ 0) but all the time it projects through at points in the half

[62] The same condition is obtained for termination at end 1.

part of the Z_r-Z_z-plane corresponding to negative (or zero) Z_r-values.

On the other hand it is allowed to cross the Z_r-axis because "c", i.e. also the ratio $\frac{c}{2a}$, can be positive, zero or negative.[63]

It is now clear that the distance between the vertex of the parabola in Fig. 18 and the Z_z-axis always is $\geq$ the distance between the vertex of the paraboloid in Fig. 17 and the Z_r-Z_z-plane, and that *the vertex of the paraboloid can very well be located below*

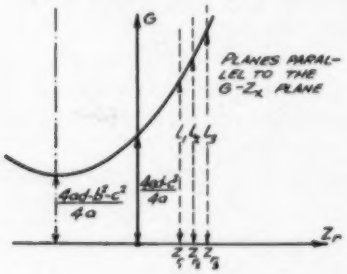


FIG. 19

this plane, but the vertex of the parabola, cut out by the G-Z_z-plane, must always be located above the Z_z-axis or at most touch it.

The vertex of each one of all other parabolas cut out by planes parallel to the G-Z_z-plane, but corresponding to positive Z_r:s, is located *higher* above the Z_z- Z_r-plane than the vertex of the parabola dealt with in the foregoing. This is readily seen from Fig. 19.

Thus to impose on the distance $\frac{4ad-b^2-c^2}{4a}$ to be ≥ 0 is incorrect, as this distance is allowed to be < 0, provided the

[63] It may be added that as "a" is shown to be ≥ 0 then "d" must also be ≥ 0 as otherwise the discriminant ($c^2 - 4ad$) cannot be ≤ 0. The same is required because G in (56) must be ≥ 0 for $Z_z = 0$. "d" *is* ≥ 0, however, because it is nothing but r_1 as shown in (52). Further from the discriminant (58) and from (56) it is clear that "c" can have any sign.

distance $\frac{4ad-c^2}{4a}$ is kept ≥ 0, *which happens to correspond to a termination in a purely reactive network.*

As l_1 l_2 l_3 etc., being distances between the vertices of an infinite set of cut out parabolas, corresponding to a termination in a general impedance network, and the $Z_r - Z_x$ plane, are $> \frac{4ad-c^2}{4a}$, we see that *a condition obtained for a pure reactance termination is the most severe possible,* and a condition obtained for a termination in a general impedance network is always satisfied when (60) is satisfied.

It may be pointed out that the condition (60) naturally holds for any value of Z_x, consequently also for short-circuit and open-circuit at the terminating end, i.e. for the four terminal network when "standing alone."

The necessary condition (60) could have been obtained in at least one more way. This would be to terminate our four-terminal network in a passive two-terminal network and impose on the active power component at the end where the e.m.f. is applied to be $\geq$ the active power component consumed by the terminating network. (For a pure reactance termination this latter component is zero.)

This alternative method, as well as the one we have used, is based upon the Law of Conservation of Energy. Our method of attack is, however, more interesting.

(**d**). Now we are going to show that, *if dealing with the* β*:s instead of the* α*:s, a necessary condition analogous to (60), but in terms of the real parts of the* β*:s, can be reached.*

Again we terminate our four-terminal network in a passive two-terminal network as shown in Fig. 16.

Notations: Let the driving-point impedance of the terminated four-terminal network and of the termination itself be denoted as under the previous item shown in (45) and (46).

Then from (43) and (22) we get:

$$Z' = \frac{E_1}{I_1} = \frac{\beta_{11}Z + (\beta_{11}\beta_{nn} - \beta_{1n}^2)}{Z + \beta_{nn}} \qquad \ldots\ldots(61)$$

Further let:

$$\left.\begin{aligned} \beta_{11} &= h_1 + j\,v_1 \\ \beta_{nn} &= h_n + j\,v_n \\ \beta_{1n} &= h_{1n} + j\,v_{1n} \end{aligned}\right\} \qquad \ldots\ldots(62)^{63*}$$

and after substitution of (45), (46) and (62) into (61) we get an expression similar to (48). When then solving for the real part we get an expression similar to (49), and when proceeding in the same manner as under the previous item we obtain the inequality:

$$-4\,(h_1h_n - h_{1n}^2)\,(h_1h_n + v_{1n}^2) \leq 0$$

The last factor can be disregarded, because it is always positive.

Thus the inequality:

$$(h_1h_n - h_{1n}^2) \geq 0, \textit{ for } \gamma \geq 0 \qquad \ldots\ldots(63)$$

(supplemented with $h_1 \geq 0$, for $\gamma \geq 0$, *or* $h_n \geq 0$, for $\gamma \geq 0$) *is a necessary condition (obtained for termination at either end) which always must be satisfied!*

As to its necessity for a general impedence termination, the same discussion as under the previous item applies.

[63*] *The author regrets not having selected a notation for the real and imaginary parts of the α:s in (47) and β:s in (62) more in accordance with standard practice for functions of the character in question, i.e. the real and imaginary parts of the α:s (being admittances) should have been denoted* g_1, g_n, g_{1n} *and* b_1, b_n, b_{1n} *respectively, while the real and imaginary parts of the β:s (being impedances) should have been denoted* r_1, r_n, r_{1n} *and* x_1, x_n, x_{1n} *respectively.*

On the other hand, in most of our deductions we are going to use the α:s and their real and imaginary parts may therewith stand out more definitely when denoted r:s and x:s than they would have done if denoted g:s and b:s respectively.

(e). Now, naturally the question may be raised *whether either condition* (60) *or condition* (63) *can be dispensed with?*

This can actually be done as shown below.

Let:

$$(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) = d + jd'$$

Then by the aid of (47) we get:

$$(r_1r_n - r_{1n}^2) - (x_1x_n - x_{1n}^2) + j(r_1x_n + r_nx_1 - 2\,r_{1n}x_{1n}) = d + jd'$$

so that if we write, for brevity:

$$(r_1r_n - r_{1n}^2) = \Delta$$

then we have:

$$(x_1x_n - x_{1n}^2) = \Delta - d$$

On the other hand, from (16) we know:

$$\left.\begin{aligned} \beta_{11} &= \frac{r_n + jx_n}{d + jd'} \\ \beta_{nn} &= \frac{r_1 + jx_1}{d + jd'} \\ \beta_{1n} &= \frac{r_{1n} + jx_{1n}}{d + jd'} \end{aligned}\right\}$$

Hence from (62) by rationalizing the denominators of above β:s we get, when solving for the real parts:

$$\left.\begin{aligned} h_1 &= \frac{r_nd + x_nd'}{d^2 + d'^2} \\ h_n &= \frac{r_1d + x_1d'}{d^2 + d'^2} \\ h_{1n} &= \frac{r_{1n}d + x_{1n}d'}{d^2 + d'^2} \end{aligned}\right\}$$

Forming the h-determinant gives:

$$(d^2 + d'^2)^2(h_1h_n - h_{1n}^2) = (r_1d + x_1d')(r_nd + x_nd') - (r_{1n}d + x_{1n}d')^2 = d^2(r_1r_n - r_{1n}^2) + d'^2(x_1x_n - x_{1n}^2) + dd'(r_1x_n + r_nx_1 - 2r_{1n}x_{1n}) = d^2\Delta + d'^2(\Delta - d) + dd'^2 = d^2\Delta + d'^2\Delta = \Delta(d^2 + d'^2)$$

i.e.

$$(h_1h_n - h_{1n}^2) = \frac{r_1r_n - r_{1n}^2}{d^2 + d'^2} \qquad \ldots\ldots(64)$$

Thus, as the denominator is a sum of two squares, we see that if $(r_1r_n - r_{1n}^2)$ *is* ≥ 0, *then* $(h_1h_n - h_{1n}^2)$ *is necessarily also* ≥ 0 *and vice versa!*

Further, note that we have in our proof not used the fact that r_1, r_n, h_1 and h_n are each ≥ 0, for $\gamma \geq 0$, i.e. we could as well have used other letters in our proof.

Consequently, we have a general theorem, to our knowledge not known before, which can be stated as follows:

THEOREM I:[64]

If the determinant of the real parts of the elements of a symmetrical, non-singular, square matrix, whose four elements are functions of a complex variable, is $\geq$ *zero, then necessarily the determinant of the real parts of the elements of another matrix which is the inverse of the first one is also* $\geq$ *zero.*

(f). *Theorem I can readily be enlarged to hold for symmetrical, non-singular, inverse, square matrices* (whose elements are functions of a complex variable) *of any order.*

The proof for this fact, mainly due to Dr. P. Franklin of the Mathematical Department of The Massachusetts Institute of Technology, is obtained by the aid of matrix and determinant

[64] For reaching this important and interesting Theorem the writer is indebted to Prof. *F. L. Hitchcock* of the Mathematical Department of the Massachusetts Institute of Technology.

theory in combination with the theory of the positive definite quadratic forms.

Being rather long the proof is not included in this paper but it is to be found in the author's doctor's dissertation.

Its validity is beyond question and thus we can state:

THEOREM II:[65]

If the determinant of the real parts of the elements of a symmetrical, non-singular square matrix of any order, whose elements are functions of a complex variable, is ≥ 0, *then the determinant of the real parts of the elements of its inverse matrix is also* ≥ 0.

(**g**). When dealing with a driving-point *impedance* of a two-terminal network, it is immediately evident that because the real part of such a function is ≥ 0, the real part of its inverse, i.e. of the driving-point *admittance*, is also ≥ 0.

In our case, however, $Re\ \beta_{11}$ and $Re\ \beta_{nn}$ will not necessarily be ≥ 0, for $\gamma \geq 0$ if only $Re\ \alpha_{11}$ and $Re\alpha_{nn}$ are ≥ 0, for $\gamma \geq 0$, and vice versa.

In addition the condition (60) *or* (63) *must be satisfied*, as shown below.

The real part of β_{11} (say) can, by the aid of (16) and (47), be found to be:

$$Re\,\beta_{11} = h_1 = \frac{(r_1 r_n - r_{1n}^2)\, r_n + r_n x_{1n}^2 + r_1 x_n^2 - 2x_n r_{1n} x_{1n}}{(r_1 r_n - r_{1n}^2 - x_1 x_n + x_{1n}^2)^2 + (r_1 x_n + r_n x_1 - 2r_{1n} x_{1n})^2}. \quad (65)$$

The denominator in the sum of two squares and can be disregarded. Thus, we only have to show that the numerator of (65), made up of both real and imaginary parts of our α:s, always is ≥ 0 when $(r_1 r_n - r_{1n}^2)$ and r_n are each ≥ 0.

(1) Subtract $2\, x_n x_{1n} \sqrt{r_1 r_n}$ from—and add the same expression to the numerator of (65), giving:

$$Re\,\beta_{11} = \frac{(r_1 r_n - r_{1n}^2) r_n + (x_{1n}\sqrt{r_n} - x_n \sqrt{r_1})^2 + 2x_n x_{1n}(\sqrt{r_1 r_n} - r_{1n})}{(-\ -\ -)^2 + (-\ -\ -)^2}$$

[65] This theorem naturally includes Theorem I.

Here the only questionable term as to positiveness is the last one. Its factor $(\sqrt{r_1 r_n} - r_{1n})$ is ≥ 0, however, because $r_1 r_n \geq r_{1n}^2$.

Further, if x_n and x_{1n} for some frequency have the *same sign*, also the last term of the above expression is positive, i.e. all terms are positive, and consequently *Re* β_{11} itself is positive or possibly zero.

Also for ω:s making x_n and x_{1n} to have *different sign Re* β_{11} is ≥ 0, because:

(2) Adding $2\, x_n x_{1n}\sqrt{r_1 r_n}$ to and subtracting it from the numerator of (65) we get:

$$Re\,\beta_{11} = \frac{(r_1 r_n - r_{1n}^2) r_n + (x_{1n}\sqrt{r_n} + x_n\sqrt{r_1})^2 - 2x_n x_{1n}(\sqrt{r_1 r_n} - r_{1n})}{(-\ -\ -)^2 + (-\ -\ -)^2}$$

Thus we see that also when x_n and x_{1n} for a certain ω have the *same sign Re* β_{11} is still ≥ 0, for $\gamma \geq 0$, if $(r_1 r_n - r_{1n}^2) \geq 0$, for $\gamma \geq 0$ and $r_n \geq 0$, for $\gamma \geq 0$.

When the r-determinant is ≥ 0, for $\gamma \geq 0$ and in addition $r_1 \geq 0$, for $\gamma \geq 0$, then we in an exactly analogous manner as shown above, can prove that *Re* β_{nn} is also ≥ 0.

Further, the same proof can be used for showing that when $(h_1 h_n - h_{1n}^2)$ is ≥ 0, for $\gamma \geq 0$ supplemented with $h_1 \geq 0$, for $\gamma \geq 0$ and $h_n \geq 0$, for $\gamma \geq 0$ respectively, then necessarily both r_n and r_1 are ≥ 0, for $\gamma \geq 0$.

Thus we can write down another theorem stating:

THEOREM III:

If in a symmetrical, non-singular, square matrix of the second order the determinant of the real parts of its elements, each being a rational, fractional function of a complex variable, is $\geq$ *zero, for* $\gamma \geq$ *zero, and when in addition the real part of the two elements which are located on its principal diagonal is each* $\geq$ *zero, for* $\gamma \geq$ *zero, then the elements of its inverse matrix are such that besides having real parts whose determinant is* $\geq$ *zero, for* $\gamma \geq$ *zero,*

the real parts of the two elements which are located on its principal diagonal are necessarily also ≥ zero, for $\gamma \geq$ zero.

On the contrary, if only the real part of each element located on the principal diagonal of a matrix, as dealt with above, is ≥ zero for $\gamma \geq$ zero, then the real part of corresponding elements of its inverse matrix need not be ≥ zero, for $\gamma \geq$ zero.

(**h**). *It is of considerable interest and usefulness, to know what our positiveness conditions under items (a) (c) and (d) mean when translated into the language of real and imaginary parts of the four General Circuit Parameters.*

Notations:

Let:

$$\left.\begin{aligned} A &= A_r + j A_x \\ B &= B_r + j B_x \\ C &= C_r + j C_x \\ D &= D_r + j D_x \end{aligned}\right\} \qquad \ldots\ldots(66)$$

Then, from (23) we get:

$$(A_r + jA_x)(D_r + jD_x) - (B_r + jB_x)(C_r + jC_x) = 1$$

i.e.

$$\left.\begin{aligned} A_rD_r - A_xD_x - B_rC_r + B_xC_x &= 1 \\ A_rD_x + A_xD_r - B_rC_x - B_xC_r &= 0 \end{aligned}\right\} \qquad \ldots\ldots(67)$$

Further, from (22) and (47):

$$\left.\begin{aligned} \alpha_{11} = r_1 + jx_1 &= \frac{D}{B} = \frac{D_r + jD_x}{B_r + jB_x} = \frac{B_rD_r + B_xD_x}{B_r^2 + B_x^2} + j\frac{(- \ -)}{(- \ -)} \\ \alpha_{nn} = r_n + jx_n &= \frac{A}{B} = \frac{A_r + jA_x}{B_r + jB_x} = \frac{A_rB_r + A_xB_x}{B_r^2 + B_x^2} + j\frac{(- \ -)}{(- \ -)} \\ \alpha_{1n} = r_{1n} + jx_{1n} &= \frac{1}{B} = \frac{1}{B_r + jB_x} = \frac{B_r}{B_r^2 + B_x^2} + j\frac{(- \ -)}{(- \ -)} \end{aligned}\right\}$$

Thus, by the aid of (67):

$$(r_1 r_n - r_{1n}^2) = \frac{B_r C_r + A_z D_z}{B_r^2 + B_z^2} = \frac{A_r D_r + B_z C_z - 1}{B_r^2 + B_z^2}$$

The denominators, being a sum of two squares, can be disregarded, and consequently we see that:

$$\left.\begin{array}{rl} r_1 \geq 0 \textit{ corresponds to}: & (B_r D_r + B_z D_z) \geq 0 \\ r_n \geq 0 \quad " \quad " & (A_r B_r + A_z B_z) \geq 0 \\ (r_1 r_n - r_{1n}^2) \geq 0 \quad " \quad " & (B_r C_r + A_z D_z) \geq 0 \\ & \textit{or } (A_r D_r + B_z C_z) \geq 1 \end{array}\right\} \ldots(68)$$

In an analogous manner we find by the aid of (22):

$$\left.\begin{aligned} \beta_{11} &= h_1 + jv_1 = \frac{A}{C} = \frac{A_r C_r + A_z C_z}{C_r^2 + C_z^2} + j\frac{(- \ -)}{(- \ -)} \\ \beta_{nn} &= h_n + jv_n = \frac{D}{C} = \frac{C_r D_r + C_z D_z}{C_r^2 + C_z^2} + j\frac{(- \ -)}{(- \ -)} \\ \beta_{1n} &= h_{1n} + jv_{1n} = \frac{1}{C} = \frac{C_r}{C_r^2 + C_z^2} + j\frac{(- \ -)}{(- \ -)} \end{aligned}\right\}$$

Thus, by the aid of (67):

$$(h_1 h_n - h_{1n}^2) = \frac{B_r C_r + A_z D_z}{C_r^2 + C_z^2} = \frac{A_r D_r + B_z C_z - 1}{C_r^2 + C_z^2}$$

Consequently:

$$\left.\begin{array}{rl} h_1 \geq 0 \textit{ corresponds to } & (A_r C_r + A_z C_z) \geq 0 \\ h_n \geq 0 \quad " \quad " & (C_r D_r + C_z D_z) \geq 0 \\ (h_1 h_n - h_{1n}^2) \geq 0 \quad " \quad " & (B_r C_r + A_z D_z) \geq 0 \\ & \textit{or } (A_r D_r + B_z C_z) \geq 1 \end{array}\right\} \ldots.(69)$$

From (68) and (69) *we note the interesting fact that* $(r_1 r_n - r_{1n}^2) \geq 0$ *and* $(h_1 h_n - h_{1n}^2) \geq 0$ *have their correspondence in the same inequality when translated into the language of the real and the*

imaginary parts of the four General Circuit Parameters, which checks Theorem I.

Further, we note that when dealing with the General Circuit Parameters both the real and the imaginary parts are coming into our positiveness conditions, while when imposing the corresponding conditions upon either our α:s or β:s the imaginary parts, so to say, vanish from the picture.

When now comparing (68) and (69) with Vaulot's conditions,[66] stated for a single frequency, we see that (68) are identical with his inequalities (4) (7) and (8), while (69) are identical with (4), (5) and (6). Due to Theorem III we now see that either (5) *or* (6) plus (7) and (8) of his conditions, or alternatively (7) *or* (8) plus (5) and (6) can be dispensed with.

With the content of this Part in mind it is now clear, that if the matrix $\left\|\begin{matrix} A & B \\ C & B \end{matrix}\right\|$ should prove to be realizable into a passive four-terminal network, then the matrices $\left\|\begin{matrix} A & C \\ B & D \end{matrix}\right\|$, $\left\|\begin{matrix} D & B \\ C & A \end{matrix}\right\|$ and $\left\|\begin{matrix} D & C \\ B & A \end{matrix}\right\|$ are also realizable:

Further, for instance, a matrix $\left\|\begin{matrix} A & CZ \\ B & \\ \hline Z & D \end{matrix}\right\|$ where Z is any driving-point function, is realizable.

[66] Page 26. The existence of Vaulot's work was revealed to the writer by Dr. W. Cauer about two weeks after having independently arrived at the conditions (60) p. 53.

Part IV

SOME FUNCTION OF A COMPLEX VARIABLE THEORY AND DERIVATION OF NECESSARY CONDITIONS IN CASE ZEROS OR (AND) POLES LIE ON THE BOUNDARY, ETC.

In the foregoing we have mainly been dealing with the whole right half of the λ-plane, and all our conditions and Theorems in items (*c*) to (*h*) of Part III are stated generally in accordance therewith.

Due to known and accepted laws in the theory of functions of a complex variable, however, it is possible to limit our efforts to the boundary (= the imaginary axis) in the λ-plane only, because neither our α:s nor β:s can have any singularities[67] to the right of said line, as shown in Part II.

We are now going to show what laws we referred to above, and begin with some function theory which is fundamental for the successful treatment of our problem.

(**a**). We must distinguish between the following two classes of functions:

(1) *Complex functions*, for instance, $(x^2 + y^2)$; $\sin(x - jy)$; etc . . ., and

(2) *Functions of a complex variable*, for instance, $f_{(z)} = f_{(x+jy)}$

The conditions which must be satisfied by a given expression $w = (u + jv)$, in which u and v signify real functions of x and y, in order that the expression may be a function of $z = (x + jy)$ are found to be:[68]

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \quad \ldots\ldots(70)$$

[67] Note: *only poles are singularities,—zeros are not.*
[68] Bibliogr. 44, p. 29.

When the conditions (70) are satisfied we say that w is an *analytic function* of z.

The two real functions u and v which satisfy the conditions (70) are called *conjugate functions*, and each of them is in German literature usually called a "*Potentialfunction*." It is an interesting fact that each one of the two conjugate functions is necessarily a solution of the Laplace differential equations in two variables, i.e:

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}&=0\\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}&=0\end{aligned}\right\} \qquad \ldots\ldots(71)$$

Conversely any real solution of the Laplace equations may be made to be the real or imaginary part of an analytic function $f_{(z)}$.

It is found[69], that the necessary and sufficient condition which must be satisfied by w in order to be a function of $z = (x + jy)$ is simply:

$$\frac{\partial w}{\partial y} = j\frac{\partial w}{\partial x} \qquad \ldots\ldots(72)$$

the real constituent parts of such a function being given by (70) and (71).

Forming $\frac{dw}{dz}$ we get:

$$\frac{dw}{dz}=\frac{du+j\,dv}{dx+j\,dy}=\frac{\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+j\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)}{dx+j\,dy}$$

and after substitution of (70) $\frac{dw}{dz}$ reduces to:

$$\frac{dw}{dz}=\frac{\partial u}{\partial x}+j\frac{\partial v}{\partial x} \qquad \ldots\ldots(73)$$

[69] Bibliogr. 44, p. 28.

i.e. $\frac{dw}{dz}$ is independent of dz and has the same value in whatever direction the infinitely small movement may take place. This is the same as to require the two complex functions w and z to fulfil the requirement (72).

The fact that the entire system of points in the w-plane definitely depends upon the system of points in the z-plane is, probably after *Riemann*, called "conformal representation," a name which nowadays usually is replaced by "conformal mapping."

Note that in the relation:

$$w^3 + w - z = 0$$

w is a *multiform* (many valued) function of z, but z is a *uniform*[70] (single valued) function of w.

(**b**). If the integral $\int f_{(z)}dz$ be extended over the boundary of a region in which we have a pole at $z = z_1$, and in such a manner that $\lim_{z \to z_1} [(z - z_1) f_{(z)}]$ approaches a definite, finite, limiting value k_1, independent of the mode of approach to z_1, then:

$$\int f_{(z)}\, dz = 2\pi j k_1$$

i.e.

$$k_1 = \frac{1}{2\pi j}\int f_{(z)}\, dz = \lim_{z \to z_1} [(z - z_1) f_{(z)}] \quad \ldots\ldots (74)$$

and k_1 *is called the* RESIDUE *of* $f_{(z)}$ *at the point* z_1.

If now $\varphi_{(z)}$ be a function which is uniform and continuous

[70] Note, our α:s and β:s are always uniform functions of λ, and thus we need not deal with any many sheeted Riemann-surface when synthesizing our four-terminal network.

A uniform function is by Cauchy called "*monodromic.*"

both in the interior, and along the boundary C, of a region T, and if t denotes an arbitrary point in this region, then the function:

$$f_{(s)} = \frac{\varphi_{(s)}}{z - t} \qquad \ldots\ldots (75)$$

has the properties required above, i.e. $f_{(s)}$ becomes singular in such a manner for $z = t$, that $[(z - t) f_{(s)}] = \varphi_{(s)}$ approaches $\varphi_{(t)}$ in the limit, as z approaches t.

Thus:

$$\int_C \frac{\varphi_{(s)}\, dz}{z - t} = 2\pi j \varphi_{(t)}$$

i.e.

$$\varphi_{(t)} = \frac{1}{2\pi j} \int_C \frac{\varphi_{(s)}\, dz}{z - t} \qquad \ldots\ldots (76)$$

This is the important and well-known *Cauchy's line integral*[n] extended along the boundary of the regular region T and bringing out the interesting fact that *the value of a function of a complex variable, regular within a certain region, is at every point of the interior determined by the values it assumes on the boundary C of this region.*

(c). If the curve C is a circle with its center at t, and if we put:

$$\varphi_{(s)} = u_{(s)} + jv_{(s)}$$

and

$$\varphi_{(t)} = u_{0(s)} + jv_{0(s)}$$

further

$$(z - t) = r(\cos\emptyset + j\sin\emptyset)$$

then we get:

$$dz = r(-\sin\emptyset + j\cos\emptyset)d\emptyset = rj(\cos\emptyset + j\sin\emptyset)d\emptyset$$

[n] Bibliogr. 44, p. 113. Bibliogr. 43, p. 196.

i.e.

$$\frac{dz}{z-t} = j\,d\emptyset$$

Consequently after subsitution into (76) we get:

$$u_{0(z)} + jv_{0(z)} = \frac{1}{2\pi j}\int_0^{2\pi}[u_{(z)} + jv_{(z)}]\,jd\emptyset = \frac{1}{2\pi}\int_0^{2\pi} u_{(z)}d\emptyset +$$

$$+ j\frac{1}{2\pi}\int_0^{2\pi} v_{(z)}\,d\emptyset$$

and finally, because z is a function of x and y:

$$\left.\begin{aligned} u_{0(xy)} &= \frac{1}{2\pi}\int_0^{2\pi} u_{(xy)}\,d\emptyset \\ v_{0(xy)} &= \frac{1}{2\pi}\int_0^{2\pi} v_{(xy)}\,d\emptyset \end{aligned}\right\} \quad \ldots\ldots(77)^{72}$$

Equations (77) tell us that the value of the real, as well as of the imaginary, part of a function of a complex variable at the center of a circle within which no singularities exist is equal to the mean of its values along the circumference, and when the radius of this circle is sufficiently small it is clear that *neither* $u_{0(xy)}$ *nor* $v_{0(xy)}$ *can have a maximum or a minimum at an inner point of the domain bounded by the circle.*

Bieberbach ([47]) has enlarged the validity of (77) and states that for any region T (not necessarily small) in which a potential function is everywhere finite and continuous (i.e. regular) any point $u_{(xy)}$ on the u-surface (say), above the z-plane can be represented with the equality:

$$u_{(xy)} = \frac{1}{2\pi}\int u_{(\xi\eta)}\,\frac{(R^2 - r^2)}{R^2 + r^2 - 2Rr\cos(\delta - \varphi)}\,d\delta \ldots.(78)^{73}$$

[72] Bibliogr. 43, p. 197.
[73] Bibliogr. 47, Band I. S. 70.

This is the important *Poisson's line integral.*
(ξ,η) are the coordinates on the boundary of T, and (xy) those in the interior.

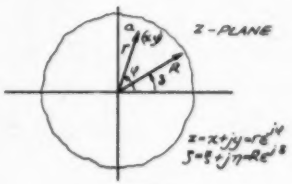


FIG. 20

If now the point "a" moves into the origin, then (78) goes over into:

$$u_{0(x_0)} = \frac{1}{2\pi}\int_0^{2\pi} u_{(\xi\eta)}\, d\delta \qquad \ldots\ldots(79)$$

i.e. we are back to the same expression as in (77), but in this case the region T is not required to be necessarily very small.

From (78) it is now clear that a *potential function*, being the real or imaginary part of a function of a complex variable, *which is regular within a certain region and also regular on the boundary surrounding this region, is characterized by a law saying that the value of the function at any point inside the boundary is falling between the extremal values on the boundary!*

This is the same as to say that *for such a function the maximum as well as minimum value must fall on the boundary!*

(**d**). This boundary, however, has interest from a purely mathematical point of view only when considered as a circle or another similar closed curve.

What we want as boundary is the imaginary axis (from $-\infty$ to $+\infty$) *in the* λ*-plane,* but it is known that all points inside a unit circle in one complex plane can be transformed to fall to the right of the imaginary axis in another (or the same) complex plane by "conformal mapping."

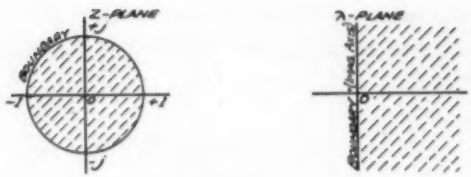


FIG. 21

A transformation which will accomplish this is obtained by letting:

$$z = \frac{\lambda - 1}{\lambda + 1} \quad \text{i.e.} \quad \lambda = \gamma + j\omega = \frac{1 + z}{1 - z}$$

Thus:

$$\gamma + j\omega = \frac{1 + x + jy}{1 - x - jy} = \frac{1 - x^2 - y^2}{(1 - x)^2 + y^2} + j\frac{2y}{(1 - x)^2 + y^2}$$

i.e.

$$\gamma = \frac{1 - x^2 - y^2}{(1 - x)^2 + y^2}; \quad \omega = \frac{2y}{(1 - x)^2 + y^2}$$

The correctness of above transformation can readily be seen by a simple check.

What in addition we would have to look for is how to get a *closed* boundary as required by (78).

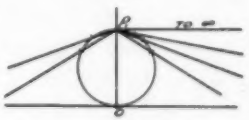


FIG. 22

It is known[74], however, that for ω-values close to infinity it is perfectly legitimate to add a half circle, and from a purely mathematical point of view *the extreme ends of the imaginary (as well as real) axis are considered joined at infinity, which can best be*

[74] Bibliogr. 7, p. 158.

seen when projecting the λ-plane steriographically upon a unit λ-sphere touching the plane at the origin, as shown in Fig. 22

On this sphere P corresponds to both ends of the imaginary axis as well as to every other point at ∞!

(e). From Part II we know that, disregarding the case when poles lie on the boundary, our α:s as well as β:s are all regular in the right half of the λ-plane, i.e. each of them fulfils there the requirements embodied in Cauchy's line integral (76) stated in terms of values along the boundary of a region within which no singularities exist. This boundary we have shown can be the imaginary axis (say) of a complex plane as well as a circle in such a plane, and consequently for each one of our functions the fact holds that every point in the interior of the right half of the λ-plane is definitely defined by the values[75] it assumes on the boundary.

In addition, and what is even more helpful, Poisson's integral (78) tells us that the real (as well as the imaginary) part of our α:s and β:s, because being defined by Cauchy's integral in the right half of the λ-plane, are potential functions having such a character that their minimum value (the maximum value can more or less be left out of consideration) falls on the boundary.

The imaginary parts do not interest us, but the real parts do, and it is clear that *if* $Re\,\alpha_{(j\omega)}$ *curve, say, is located above the boundary for all real* ω*:s, then* $Re\alpha$*-sheet at no point dips below the λ-plane on the right side of the boundary*; and more than that, the sheet does not even touch the λ-plane within the region in question, because this would require $Re\alpha_{(j\omega)}$ to become negative for some particular ω.

[75] Note, Cauchy's integral, when applied to our α:s and β:s, is referring to the λ-functions as a whole, i.e., to a "*complex surface*" above the right half of the λ-plane. It is difficult to picture such a surface, however, because four dimensions would be required, if not seeing it as consisting of two independent sheets which thus has nothing in common with a two-sheeted Riemann-surface. Thus, when considering our α:s, say, every point in the λ-plane would correspond to one point on the Reα-sheet and one point on the Imagα-sheet. Then the total α is found from (Re$\alpha + j$ Imagα).

Thus it is clear that equations (42),[76] together with the regularity clause, strictly forbids zeros to fall to the right of the boundary, i.e. *zeros and poles of our driving point functions must fall in the same half of the* λ*-plane*, but note the reverse is not true, i.e. if zeros and poles have the required location, then the real part along the boundary for the corresponding function can very well take on negative (as well as positive) values on the boundary, as shown in *Example 4*.[77]

On the other hand, because $Re\ \alpha_{1n\,(j\omega)}$ and $Re\ \beta_{1n\,(j\omega)}$ are allowed to take on negative (as well as positive) values on the boundary, α_{1n} and β_{1n}, as well as $Re\ \alpha_{1n}$- and $Re\ \beta_{1n}$-sheets, are allowed to have zeros in the right half of the plane, but note, a λ-function whose real part is negative on the boundary for some or all real ω-values, may[78] or may not[79] have a zero to the right of the boundary.

If negative somewhere on the boundary, then the real part certainly must have a zero to the right of the boundary, but the imaginary[80] part need not be zero at the same point.

In the reverse sense the relation is unique, i.e. if our transfer function has a zero in the right half of the λ-plane, then its real part along the boundary must take on both positive and negative values.

The curves of *Example 2* are interesting for at least three reasons.

(1) They show that the positiveness requirement, imposed on the real part of the driving-point functions, need not be fulfilled by the real part of the transfer function.

[76] P. 46.

[77] Appendix, p. 201.

[78] See α_{1n} in *Example 2* (Appendix, p. 200).

[79] See *Example 4* (Appendix, p. 201). Note also that by substituting $\lambda = -\lambda$ in $f_{(\lambda)}$ zeros and poles are moved from the left half to the right half of the λ-plane without affecting Re $f_{(j\omega)}$. This was earlier pointed out by Brune.

[80] The imaginary part is *always* both positive and negative along the boundary because it is an *odd* potential function of ω, and when considering any one of our α:s or β:s the imaginary part of each has always a zero somewhere in the right half of the plane. The fact that the *real part is an even*—and the *imaginary part an odd potential function* will be taken up later. The latter especially is of utmost importance for a successful treatment of our problem, as shown in Part XI.

(2) They tell us that if any value of γ and any real value of ω be substituted in the r:s, considered over the whole right half of the λ-plane, then the numerical values of our r:s thus obtained always fall between the absolute $r_{min.}$ and $r_{max.}$ respectively as shown by the curves, which are plotted along the boundary.

(3) If the real part of any one of our driving-point functions has a zero on the boundary (the only place in the right half of the λ-plane where it can happen), then the real part of the transfer function must have a zero at the same point, as otherwise the condition that the determinant of the real parts be $\geq$ zero would be violated.

As mentioned on p. 69 a λ-function falling under the Cauchy's integral clause has to be regular not only in a certain region but also on the boundary surrounding this region. On the other hand we know from Part II that zeros and poles of our three functions very well can fall on the imaginary axis, which is our boundary.

All such points[81] require special attention due to the fact that a function of a complex variable attains different values close to the point in question, depending upon the path along which it is approached.

(f). *Let us first take up the case of a zero on the boundary* at a point "a" $= j\sigma_1$.

Naturally, provided "a" is not located at the origin or at ∞, we then have *two* zeros, at $\pm j\sigma_1$, as they always appear in conjugate pairs, but only the one corresponding to a positive frequency (say) need to be considered, because it is easy to prove that the function around two conjugate points has the same character.

It is known from a *Lemma*[82] in mathematics that *a zero of multiplicity n is surrounded by 2n sectors (see Fig. 22a) of equal angles, in which the real part of the λ-function is alternately positive and negative.*

[81] In reality only points where poles lie, as zeros are not excluded by Cauchy's line integral.

[82] Bibliogr. 38, p. 28.

As the real part of our driving-point functions has to be positive to the right of the boundary, we see that of such a function, a zero *on* the boundary cannot have more than two sectors, each of 180 degrees.[83]

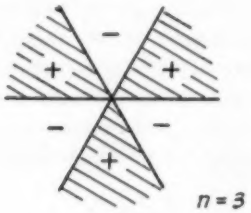


FIG. 22*a*

Consequently such a zero must be simple!

If now the zero is surrounded by a circle of small radius (bounding the *domain* of "a"), then it can be shown that the function $\varphi_{(t)}$ inside this circle (fulfilling the requirement of Cauchy's integral) can be expanded in a Taylor's series,[84] obtained by expanding $\frac{1}{z-t}$ in a convergent series to be substituted into (76).

Thus we get:

$$\varphi_{(t)} = \varphi_{(a)} + (t-a)\,\varphi'_{(a)} + \frac{(t-a)^2}{2}\varphi''_{(a)} + \ldots \qquad \ldots\ldots(80)$$

where $\varphi'_{(a)}$, $\varphi''_{(a)}$, etc. are the corresponding derivatives of $\varphi_{(t)}$ at "a".

As the distance between "t," *which is any point inside the circle*, and "a" becomes very small, all terms of higher degree than the first can be neglected.

Further, when letting:

$$(t-a) = re^{j\phi}$$

[83] See Fig. 23, p. 88; the division line, however, must fall along the imaginary axis.

[84] Bibliogr. 44, p. 118.

and

$$\varphi'_{(a)} = \kappa e^{j\psi}$$

(where r and κ are real constants). Then we get:

$$\varphi_{(t)} = \varphi_{(a)} + r\kappa e^{j(\phi+\psi)} = \varphi_{(a)} + r\kappa\,[\cos(\phi+\psi) + j\sin(\phi+\psi)]$$

$\varphi_{(a)}$ is equal to zero, and as we are interested in only the real part of $\varphi_{(t)}$, we get:

$$Re\,\varphi_{(t)} = r\kappa\cos(\phi+\psi) \qquad \ldots\ldots(81)$$

For our driving-point functions this expression has to be real and $\geq$ zero, for $-\frac{\pi}{2} < \phi < +\frac{\pi}{2}$.

For this to be true ψ must be zero, and thus we see that the only requirement is that the differential coefficient $\varphi'_{(a)}$ of these functions, i.e. $\left[\frac{d}{d\lambda}\alpha_{11}\right]_{\lambda=a}$, $\left[\frac{d}{d\lambda}\alpha_{nn}\right]_{\lambda=a}$, $\left[\frac{d}{d\lambda}\beta_{11}\right]_{\lambda=a}$ and $\left[\frac{d}{d\lambda}\beta_{nn}\right]_{\lambda=a}$ respectively *be a positive, real, non-zero constant!*

Why $\varphi'_{(a)}$ has to be "non-zero" is evident because for $(t-a)$ approaching zero we would otherwise get zero times zero, i.e. a double zero, which is not allowed from what we said about a zero (of the above functions) on the boundary.

As to our transfer functions, α_{1n} or β_{1n}, the requirements are not so strict, because the real part of these functions need to be $\geq$ zero neither to the right of $-$ nor on the boundary itself.

Therefore, zeros on the boundary of these functions are not required to be simple, and further, it is clear that the differential coefficient $\varphi'_{(a)}$ only needs to be zero, or a real constant $<$ or $>$ zero. *If $\neq$ zero the absolute value of $\varphi'_{(a)}$, however, is for the transfer function limited by a determinant condition, to be obtained under item (h).*

(**g**). *Let us now proceed to the case of a pole on the boundary*, at a point "b" = $j\sigma_2$ (i.e. generally two conjugate poles at $\pm j\sigma_2$).

As a pole of a function is nothing but a zero of its reciprocal,

we see that *for each one of our functions* (all having generally the same denominator) *a pole* (*like a zero*) *on the boundary must be simple*!

In the case of a pole the function is no longer finite and continuous at the point in question, but becomes infinite (of a finite integral order) when considered infinitesmally close to "b."

The reciprocal of the function, however, is zero and absolute continuous at "b" and close to it, and consequently the same discussion as for a zero on the boundary is valid, when here considering $\left[\frac{d}{d\lambda}\frac{1}{\alpha}\right]_{\lambda=b}$ *or* $\left[\frac{d}{d\lambda}\frac{1}{\beta}\right]_{\lambda=b}$.

Instead of working with the derivative of the reciprocal, however, it is more convenient to deal with the *residue* of the function as it stands, because the logarithmic derivative of a function of a complex variable is equal to minus the quotient of the function itself and its residue, at a simple pole; in other words, *the derivative of* $\frac{1}{f_{(\lambda)}}$ *say, at a simple zero on the boundary is simply equal to the reciprocal of the residue of* $f_{(\lambda)}$ *at corresponding simple pole*!

Because the function at a pole becomes singular we cannot expand it inside a small circle in a Taylor's series, as in the case of a zero on the boundary, but must adopt a Laurent's series[85] which is a combination of a Taylor's series and another series of descending powers of t, both obtained by taking Cauchy's line integral along *two* concentric circles around the pole.

If we assume that the pole is located at the origin and if we denote the differential coefficients of Taylor's series by $q'\ q''\ q'''$ . . . etc., then we get:

$$f_{(t)} = q^0 + q't + \frac{q''t^2}{2} + \ldots\ldots + \frac{k_0}{t} + \frac{k_2}{t^2} + \frac{k_3}{t^3} + \ldots\ldots \qquad \ldots\ldots (82)$$

where k_0 is the residue at the pole.

[85] Bibliogr. 43, p. 247.

As the pole is simple, equation (82) simplifies into:

$$f_{(t)} = \frac{k_0}{t} + q^0 + q't + q''\frac{t^2}{2} + \dots \qquad \dots\dots(83)$$

The derivative of $\frac{1}{f_{(t)}}$ is $= -\frac{f'(t)}{[f_{(t)}]^2}$

Forming this quotient from the derivative and square respectively of $f_{(t)}$ in (83) we get:

$$\frac{f'_{(t)}}{[f_{(t)}]^2} = \frac{\frac{-k_0}{t^2} + q' + \dots\dots}{\frac{k_0^2}{t^2} + \frac{2q^0k_0}{t} + \dots\dots} = \frac{-k_0 + q't^2 + \dots.}{k_0^2 + 2q^0k_0t + \dots.}$$

which in the limit for $t = 0$ goes over into:

$$-\frac{f'_{(0)}}{[f_{(0)}]^2} = \frac{1}{k_0} \qquad \dots\dots(84)$$

i.e. the statement as to the relation between the derivative of the reciprocal of the function and the residue of the function as it stands, holds!

Now our functions can, between two small circles around $b = j\sigma_2$, be expanded in a series similar to (83) i.e.

$$\varphi_{(t)} = \frac{k_b}{(t-b)} + q + q'(t-b) + q''\frac{(t-b)^2}{2} + \dots. \qquad \dots(85)$$

which for t-points close to b, reduces to:

$$\varphi_{(t)} = \frac{k_b}{(t-b)}$$

Letting: $(t-b) = re^{j\phi}$ and $k_b = K_b e^{j\psi}$ (where r and K_b are real constants), then we get:

$$\varphi_{(t)} = \frac{K_b}{r}e^{j(\psi-\phi)} = \frac{K_b}{r}[\cos(\psi-\phi) + j\sin(\psi-\phi)]$$

the real part of which is:

$$Re\,\varphi_{(i)} = \frac{K_b}{r}\cos(\psi - \phi) \qquad \ldots\ldots(86)$$

For our driving-point functions this expression has to be real and $\geq$ zero, for $-\frac{\pi}{2} < \phi < +\frac{\pi}{2}$.

Consequently ψ must be zero, and thus the only requirement is that the residue of these functions, i.e. $\lim_{\lambda \to b}[(\lambda - b)\,\alpha_{11}]$; $\lim_{\lambda \to b}[(\lambda - b)\,\alpha_{nn}]$; $\lim_{\lambda \to b}[(\lambda - b)\,\beta_{11}]$; and $\lim_{\lambda \to b}[(\lambda - b)\,\beta_{nn}]$ respectively, be a *positive, finite, real constant!*

The transfer function again makes an exception, as its residue for a pole on the boundary may be positive or negative, but always real and finite.

Like the absolute value of the differential coefficient of this function also *the absolute value of its residue is limited by a determinant condition to be shown under item* (*i*).

(**h**). *Let us now assume that all three of our functions* (α:s or β:s) *have a common zero* at a point "a" $= j\sigma_1$, (i.e. $\pm j\sigma_1$) *on the boundary.*

Expanding each one of the functions inside a small circle surrounding "a," in a Taylor's series we reach (81), the discusion of which was given under item (f). Besides equation (81), deduced from a consideration of the real part of a single function, *there must, however, exist a determinant condition* to be deduced by the aid of our condition (60) or (63) respectively, obtained in items (c) and (d) of Part III.

Letting:

the differential coeff. of α_{11} *at* "a" *be* $\alpha'_{11(a)}$
" " " " α_{nn} " " " $\alpha'_{nn(a)}$
" " " " α_{1n} " " " $\alpha'_{1n(a)}$

then, from item (f), we know that all $\alpha'_{(a)}$:s are real, and we have:

$$\left.\begin{aligned} r_1 &= r\,\alpha'_{11(a)}\,\cos\phi \\ r_n &= r\,\alpha'_{nn(a)}\,\cos\phi \\ r_{1n} &= r\,\alpha'_{1n(a)}\,\cos\phi \end{aligned}\right\}$$

i.e. the determinant condition (60) gives:

$$r\,\alpha'_{11(a)}\cos\phi\; r\alpha'_{nn(a)}\cos\phi - [r\alpha'_{1n(a)}\cos\phi]^2 = [\alpha'_{11(a)}\,\alpha'_{nn(a)} - \alpha'^2_{1n(a)}]\,r^2\cos^2\phi \geq 0 \quad \ldots.(87)$$

$r^2\cos^2\phi$ in (87) is always real and ≥ 0. Consequently, the necessary condition we have arrived at is:

$$[\alpha'_{11(a)}\,\alpha'_{nn(a)} - \alpha'^2_{1n(a)}]\ \textit{must be real and} \geq 0 \quad ..(88)$$

i.e. in words:—

At a common zero on the boundary the determinant of the three differential coefficients of our functions must always be real and $\geq$ *zero!*

(i). *Let us now proceed to consider the case of a common pole on the boundary.*

Again we can expect to reach a determinant condition, this time imposed on the residues of our functions at the point in question,[88] and that is what we actually find, as shown below.

Letting the residues of our α:s (say) at the common pole be k_1 k_n and k_{1n} respectively, and expanding each separate α into a Laurent's series, as shown in (85), then, knowing the fact that the k:s are real, we get from (86) infinitesimally close to the pole:

$$\left.\begin{aligned} r_1 &= \frac{k_1}{r}\cos\phi \\ r_n &= \frac{k_n}{r}\cos\phi \\ r_{1n} &= \frac{k_{1n}}{r}\cos\phi \end{aligned}\right\}$$

[88] This was predicted by Brune.

i.e. from the determinant condition (60) we get:

$$(k_1 k_n - k_{1n}^2)\frac{\cos^2\phi}{r^2} \geq 0 \qquad \ldots.(89)$$

$\frac{\cos^2\phi}{r^2}$ in (89) is always real and ≥ 0, i.e. we have the necessary condition:

$$(k_1 k_n - k^2{}_{1n}) \textit{ must be real and } \geq 0 \qquad \ldots\ldots(90)$$

which in words says:

At a common pole on the boundary the determinant of the three residues of our functions must always be real and $\geq$ zero!

(j). *As to the occurrence of zeros or poles on the boundary, they cannot be contained in our functions in an arbitrary manner.* So for instance (considering the α:s say), due to (60), if α_{11} (and not α_{nn} or vice verse) has a *zero* on the boundary, then α_{1n} must have a zero, or be purely imaginary, at the same point.[87]

The condition (60) is even more certainly satisfied if only α_{1n} but not α_{11} and α_{nn} have a zero at "a," say.

If all three α:s have a common zero at "a," which very seldom happens, then condition (88) must be satisfied.

It might be added that in case of a zero of α_{1n} in the right half of the λ-plane, the real part of α_{11} and of α_{nn} are very much bigger than the real part of α_{1n} infinitesmally close to the zero, i.e. we are again safe.

The occurrence of *poles* on the boundary is quite different from that of zeros. So for instance, if one of our functions has a pole there, then all three have practically always a pole at the same point, because they generally have the same denominator as seen from their definitions,[88] although exceptions occur[89] due to different character of the minors.

[87] This must also be true if both α_{11} and α_{nn} contain the same zero; the condition (88) must be satisfied, however. Note, if the point in question is the origin or ∞, then α_{1n} *must* have a zero there, if α_{11} or α_{nn} or both have a zero there.

[88] Pp. 29–30.

[89] See *Example 2* (Appendix, p. 200) where α_{11} only has a pole at the origin.

When dealing with residues at a point where not all functions have a pole, it can be shown that *the residue of the function which has no pole at the point in question is zero!*

Thus from (90) we see that *neither can* α_{n1} *alone have a pole at a given point on the boundary, nor can only* α_{1n} *and* α_{11}, *nor* α_{1n} *and* α_{nn}. On the other hand, *nothing is violated if* α_{11} *or* α_{nn} *alone have a pole at a given point on the boundary, or both* α_{11} *and* α_{nn} *but not* α_{1n}.

(k). *The above statement may be clearer when considering the facts given below*:

We have no means for removal of zeros from the boundary, while poles located there can be removed by a simple partial fraction expansion, where the constant coefficient in the numerator of the removed pole(s) is nothing but the corresponding residue of the function (for a pole at the origin or at ∞) or twice the residue (for a conjugate pair of poles at a finite ω).

When dealing with a two-terminal network, which is specified by a single function, and saying that we remove zeros from the boundary, we in reality remove poles from the reciprocal of the function, i.e. if such a zero is contained in the impedance function we take its reciprocal and remove a pole from the admittance function, and vice versa.

In our case the "reciprocal" of our α:s are our β:s and vice versa, but from the corresponding relations[90] we see that only when all three α:s (say) have a common zero on the boundary, can it be removed as a pole from the β:s, and if α_{1n} together with either α_{11} or α_{nn} have a zero at a common point, a pole can be removed from the β_{11} or β_{nn} respectively. If α_{1n} only has a zero on the boundary it does not result in a removal of a corresponding pole from any one of the β:s.

Considering the physical side, it will be seen that separation of poles in the above manner always corresponds to the determination of elements in corresponding network.

[90] Equations (15) and (16), p. 31.

Poles, at the same point on the boundary, removed from either one of or the two driving-point functions only, or from all three functions, are realized into purely reactive structures.

A removal of a pole from the transfer function alone[91], or from this function plus either one of the two driving-point functions, would be absurd, because such removed poles have no physical meaning and cannot be realized. *It is for this reason that they are not allowed to occur.*

Note, a zero on the boundary shows up in the real part of corresponding function, while the real part of the function before and after poles on the boundary have been removed is the same.[92]

(1). *Finally a few words will be added as to zeros and poles at the mathematical point "infinity;"* how to find the differential coefficient of a function having a zero at infinity or the residue for a pole there, and how to remove such a pole from the function.

We know from our discussion of the λ-sphere that infinity is a "point" on the boundary, and it would seem evident that the differential coefficient of a function $f_{(\lambda)}$, having a zero at ∞, should be obtained from $\left[\frac{d}{d\lambda} f_{(\lambda)}\right]_{\lambda=\infty}$ and the residue from $\lim_{\lambda\to\infty}[(\lambda - \infty) f_{(\lambda)}]$.

This is, however, not quite so, which is evident from the fact that the above differential coefficient comes out equal to zero and the residue becomes undetermined, if not infinite, even if $f_{(\lambda)}$ *is* an impedance- or an admittance function of a passive network,[93] which does not check our conditions arrived at in the foregoing.

We must, on the other hand, remember that "infinity" is an '*invention*" (a really helpful one indeed) of mathematicians, and

[91] See *Example 5* (Appendix, p. 202).
[92] See Bibliogr. 38, p. 32, and *Example 5* (Appendix, p. 202).
[93] See *Example 6* (Appendix, p. 203).

everything comes out correct if in our function we substitute $\frac{1}{\lambda}$, for λ, and then consider the origin instead of ∞.

Thus if $f_{(\lambda)}$ has a zero at ∞, we find our differential coefficient from:

$$\left[\frac{d}{d\lambda'} f_{\left(\frac{1}{\lambda'}\right)}\right]_{\lambda'=0}$$

and our residue from:

$$\lim_{\lambda' \to 0} \left[\lambda' f_{\left(\frac{1}{\lambda'}\right)}\right]$$

An even simpler way of finding the residue of a function at a pole at ∞ is arraigning for descending power in λ and dividing the numerator of the function by its denominator. Then in the quotient the coefficient of the highest (being the only one we need to solve for) power of λ is the residue in question.

(**m**). With all items of this Part in mind, *it is now clear that in the following instead of considering our functions over the whole right half of the* λ*-plane*, which procedure often is very laborious, if not impossible, to handle, *we can limit our efforts to the boundary* and if poles happen to lie there they have to be given special attention.

Several of our proofs in the following, however, are, for the sake of generality, stated in terms of both γ and ω, but then they also include the boundary, being a special case for $\gamma = 0$.

The notations mentioned on p. 48 (footnote 57) will tell us what we consider.

PART V

DEFINITIONS, AND SOME THEOREMS ON "POSITIVE REAL" MATRICES, MAINLY BASED UPON THE CONTENT OF THE PRECEDING PART, ETC.

In the foregoing we have mainly dealt with *functions*, but in the Parts to follow we are gradually *applying the properties of these functions to our problem*, and this application is new.

Then, as we often shall have to mention the procedure of giving a set of functions, satisfying certain necessary conditions, a physical interpretation in form of a passive network, we shall adopt some definitions and names to be stated in the following and used throughout.

(**a**). Let us first adopt two of Brune's definitions,[94] being very expressive, and add some modifications and supplements.

Definition: If a function can be an impedance- or an admittance function of a passive network, then this function is said to have a "*network representation*," and the process of synthesizing such a network from a prescribed function is called "*finding a network representation*" of the *function*.

Further, let us in an analogous manner speak of "*finding a network representation*" of a *matrix* whose elements are the two driving-point functions and the transfer function, i.e. of a matrix whose elements are either our three α:s or β:s.

(**b**). *Definition:* A rational, fractional function $f_{(\lambda)}$, which is real, for λ real, is a "real" function, and if in addition the real part

[94] Bibliogr. 38, p. 25. Another way in which Brune, by the aid of the *Schwarzian Lemma* and *Pick's Theorem*, defines a "positive real" function is that its radius vector always lies closer to the reference line than the radius vector of λ, provided the latter lies within the angle $-\frac{\pi}{2} < \arg.\,\lambda < +\frac{\pi}{2}$, i.e. if: $|\arg.\,f_{(\lambda)}| \leq |\arg.\,\lambda|$ when $|\arg.\,\lambda| \leq \frac{\pi}{2}$, then $f_{(\lambda)}$ is a "positive real" function. See Bibliogr. 38, pp. 34–37.

of $f_{(\lambda)}$ is $\geq$ zero, for $\gamma \geq$ zero, then $f_{(\lambda)}$ is called a "*positive real*" function.[94]

(c). *Definition*: A symmetrical square matrix of the second order, whose elements $f_{11(\lambda)}, f_{nn(\lambda)}$ and $f_{1n(\lambda)}$ are all real functions, is a "real" matrix:

$$\begin{Vmatrix} f_{11(\lambda)} & f_{1n(\lambda)} \\ f_{1n(\lambda)} & f_{nn(\lambda)} \end{Vmatrix}$$

If in addition the two elements which are located on the principal diagonal have such a property, that the real part of each is ≥ 0, for $\gamma \geq 0$, and if further the determinant of the real parts of the elements, i.e.

$$Re\, f_{11(\lambda)}\, Re\, f_{nn(\lambda)} - [Re\, f_{1n(\lambda)}]^2$$

is ≥ 0, for $\gamma \geq 0$, then let us call such a matrix *a "positive real" matrix*!

In Part VIII together with Part XI it will be proved that every "positive real" matrix, whose elements have a finite number of zeros and poles, may be regarded as an α- or β-matrix, characterizing a passive four-terminal network. This we shall in the meantime frequently anticipate as a help in our mathematical deductions.

(d). *Below we are going to state some theorems on "positive real" matrices* to be useful in the following, and all necessary conditions, not specified by these Theorems, can be dispensed with, if we can prove—*which we can*—that every "positive real" matrix has a network representation in a passive four-terminal network. Thus, from the foregoing we know that our three prescribed functions satisfy the necessary and sufficient conditions of being elements of a "positive real" matrix, but we do not know yet if a "positive real" matrix fulfils all requirements of an actual α- or β- matrix.

From the reality clause and from Theorem III[95] we know that if three prescribed functions fulfil the requirements of being elements of a non-singular "positive real" matrix, then the two elements on the principal diagonal of its inverse matrix are "positive real" functions. Further, Theorem I[96] tells us that also for the inverse matrix the determinant of the real parts is ≥ 0, for $\gamma \geq 0$. Consequently, knowing that the actual α- and β-matrices usually are two inverse square matrices, above facts can be summarized into a Theorem, closely related to the Theorems I, II and III:

THEOREM IV:

If three prescribed functions (α:*s say*)[97] *are elements of a non-singular "positive real" matrix, then necessarily its inverse matrix* (having as elements the β:s) *is also a "positive real" matrix.*

Further when considering the reciprocal of the frequency, i.e. the wave length, we evidently get:[98]

THEOREM IV, *Corollary 1*:

If $\| \alpha_{ij(\lambda)} \|$, *say, is a "positive real" matrix, then, besides* $\| \beta_{ij(\lambda)} \|$, *also* $\| \alpha_{ij(\frac{1}{\lambda})} \|$ *and* $\| \beta_{ij(\frac{1}{\lambda})} \|$ *are "positive real" matrices.*

The physical interpretation of the Theorem (provided every non-singular "positive real" matrix can be proved to be realizable) is:

Every passive four-terminal network is uniquely characterized either by a family of three short-circuit admittances *or* a family of three open-circuit impedances.

[95] P. 60.
[96] P. 58.
[97] Note: strictly we ought not to call the prescribed functions α:s or β:s, because these letters, from our definitions, refer to *actual* short-circuit admittances and open-circuit impedances, which the prescribed functions may or may not be. The letters will stand for functions derived from a physical structure as well as for prescribed functions supposed to be a family of actual admittances or impedances, however, and the neighboring text will clearly indicate in what sense the letters have to be taken.
[98] See and compare Bibliogr. 38, p. 27.

Another interpretation would be: To every passive four-terminal network corresponds a "reciprocal" four-terminal network, whose open-circuit impedance characteristics are the short-circuit admittance characteristics of the first network.

From Foster's (20) Reactance Theorem we know that: *zeros and poles of a "positive real" function, characterizing a purely reactive two-terminal network, must all fall on the imaginary axis and there have separation property.*

With the aid of Brune's Theorem V, with its Corollary[99], the above conditions can as well be stated either as:

(i) *Zeros and poles must lie on the imaginary axis.*
(ii) *The residue of the function at the poles must each be a positive, real, finite constant.*

or as:

(i) *Zeros and poles must fall on the imaginary axis.*
(ii) *The differential coefficient of a function at the zeros must each be a positive, real, non-zero constant.*

When dealing with a matrix instead of a single function, and knowing that actual α:s or β:s fulfil all requirements of being elements of a "positive real" matrix, we can by the aid of the material contained in the preceding Part state another Theorem which in itself includes the case when all zeros and poles lie on the imaginary axis, being equivalent to saying that the real part of each matrix element is zero on the imaginary axis.

It can be found that the conditions imposed upon the differential coefficients, separately and upon their determinant, are certainly necessary but they can be dispensed with, *because they are all automatically satisfied when the conditions, to be summarized in the Theorem, are satisfied.*

This is due to the fact that for a driving-point function zeros (as well as poles) on the boundary have to be simple, and as the real part of such a function is ≥ 0 along the boundary, the division line between the two real part sectors, shown in fig. 23, must be directed along the imaginary axis. Further,

[99] Bibliogr. 38, p. 33.

the positive section is falling to the right, because otherwise the "minimum" would fall in the interior of the plane instead of on the boundary which would be a contradiction to the requirements of the Poisson's line integral. In addition, it would require a zero in the interior of the right half of the λ-plane, but, as we know, such a zero is actually not present.

FIG. 23

Should the division line fall as indicated in the Figure, it would show up in $Re\, f_{(j\omega)}$.

The determinant of the differential coefficients can be dispensed with by the same way of reasoning, and the Theorem we have in mind would be:

THEOREM V:

If, besides being real for λ *real, the prescribed functions* (α:s *say*) *are such that*:

(i) *no poles lie to the right of the imaginary axis in the* λ*-plane.*

(ii) *at poles on the imaginary axis* α_{11} *and* α_{nn} *have each positive, real, finite residues.*

(iii) *the determinant of the residues at a common pole on the imaginary axis, i.e.* $(k_1k_n - k_{1n}^2)$ *is real and* ≥ 0.

(iv) $Re\, \alpha_{11(j\omega)} \geq 0$ *and* $Re\, \alpha_{nn(j\omega)} \geq 0$, *for all real* ω:s.

(v) $Re\, \alpha_{11(j\omega)}\, Re\, \alpha_{nn(j\omega)} - [Re\, \alpha_{1n(j\omega)}]^2 \geq 0$, *for all real* ω:s.

Then $\| \alpha_{ij} \|$ *is a "positive real" matrix!*

In particular the zeros of the α:s will have their proper location, and the differential coefficients for zeros on the boundary will satisfy the stipulated necessary conditions.

Note: Theorem V in one sense really contains too much, because only one of the two driving-point functions needs to be mentioned under (ii) and (iv), the other one being taken care of by (iii) and (v) respectively.

As the Theorem is written, it includes in itself all possible cases as to the character of the prescribed functions, however, i.e. when in the first place all three functions are real for λ real, and in addition:-

(1) Neither zeros nor poles lie on the boundary: Then (i), (iv)[100] and (v) only need to be considered.

(2) Some[101] of the zeros but no poles lie on the boundary: Still only (i), (iv)[102] and (v) need to be considered, as zeros on the boundary do no harm. The whole right half of the plane, including the boundary, still falls under the Poisson's line integral clause, and an investigation as to the character of the differential coefficients at the zeros in question is not needed.

(3) Some [103] or no zeros and some poles [103] lie on the boundary: Then all five items of the Theorem must be considered.[104]

(4) All zeros and all poles lie on the boundary. Then (i), (iv) and (v) are automatically satisfied[105] and we need to consider items (ii) and (iii) only.

[100] Either one of the two inequalities suffices.
[101] The rest fall in the interior of the plane within allowed regions.
[102] Same remark as footnote 100.
[103] Same remark as footnote 101.
[104] Again either of the two inequalities of (iv) and (ii) suffices.
[105] The two latter with their equal sign.

PART VI

ANALYSIS AND SYNTHESIS OF SOME STANDARD STRUCTURES, SHOWING FOR WHICH TYPES OF "POSITIVE REAL" MATRICES THEY CAN BE USED AS A NETWORK REPRESENTATION AND TO WHICH FAMILY OF FUNCTIONS THEY "BELONG," ETC.

The idea of building four-terminal networks by the aid of standard structures, which are a combination of a number of two-terminal networks connected in a certain manner, is old. The method has never been made general, however, (and probably it cannot be done either), i.e. *not every "positive real" matrix which certainly has a network representation, obtainable in some other way, can be directly realized by the use of a standard structure.*

It is true that, for instance, the unsymmetrical lattice structure unquestionably is able to take care of a very wide range of prescribed functions, but unfortunately we can obtain three independent equations only, while these equations contain four unknowns, each being the impedance- or admittance function of one of the two-terminal networks, contained in the structure.

Thus there is always something undetermined, as will be shown under item (h).

The same is true as to the unsymmetrical bridged T-structure, merely touched upon under item (g).

On the other hand, even if one of the standard structures cannot directly take care of a family of prescribed functions, satisfying necessary and sufficient conditions[105] for having a network representation, we cannot do without them, (at least most of them), because they will be used as component structures: for the realization of sets of removed common poles when applying our

[105] To be stated in this paper.

general methods; for the realization of the constants which remain after prescribed λ-functions are exhausted of all their zeros and poles,[107] and for the realization of functions between which certain relations hold, etc.

The most useful are the *L*-structures, including their special forms corresponding to the General Circuit Parameter B or C being zero, the transformer, and the symmetrical lattice-structure, although also the π- and the *T*-structures are extremely useful, only we know to which type of prescribed functions they "belong."[108]

Most of the structures can equally well be treated in terms of the α:s or the β:s. The driving-point- and transfer functions will first be derived from the structure under consideration, and then, when solving for the admittances or impedances of the two-terminal networks, contained in the four-terminal network, in terms of said functions, we can see for which range the structure in question can be used as a network representation.

As a rule the two-terminal networks are denoted as admittances $(Y\text{:s})$[109] if the α:s are considered, and as impedances $(Z\text{:s})$[109] if the β:s are considered.[110] On the other hand the character of a structure tells us whether its parts should be denoted as admittances or impedances.

Note, the α:s and β:s are in every case obtained by inspection only. This, besides being safer, is much more simple than deriving our functions from Kirchhoff's circuit equations by the aid of Cramer's Rule.[111]

[107] The total network representation of a family of prescribed functions always consists of a combination of standard structures, as will be shown in the following.

[108] When we say that a particular structure "belongs" to the α:s, say, we mean either that the structure in question cannot be characterized by a β-matrix, as in the case to be dealt with in item (c), *or* the range of prescribed functions taken care of by a structure can readily be seen when dealing with the α:s, say, but cannot be seen when dealing with the β:s, as will be shown in item (f). The reverse will be shown in items (d) and (e) respectively.

[109] Note the Y:s and Z:s should have a subscript (λ), but this will be left out for the sake of simplicity.

[110] If this rule should not be followed we get, for instance, when dealing with the unsymmetrical lattice-structure, expressions which become practically impossible to handle.

[111] In this Part we are going to deal with the whole right half of the λ-plane, be-

(a). *Let us start with the L-structure shown in Fig. 24.*

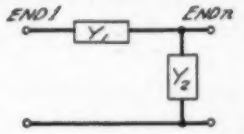


FIG. 24

When short-circuiting the two terminals at end n and applying a unit voltage of any frequency at end 1, it is evident that the current entering at this latter end is the same as the current flowing in the short-circuit, whose admittance is infinitely high in comparison to Y_2, i.e. α_{11} is $= \alpha_{1n}$.

Thus we get:

$$\left.\begin{aligned} \alpha_{11} &= \alpha_{1n} = Y_1 \\ \alpha_{nn} &= Y_1 + Y_2 \end{aligned}\right\} \qquad \ldots\ldots(91)$$

and when solving for the Y:s, we get by the aid of (47)[112]

$$\left.\begin{aligned} Y_1 &= \alpha_{11} = r_1 + jx_1 \\ Y_2 &= \alpha_{nn} - \alpha_{11} = (r_n - r_1) + j(x_n - x_1) \end{aligned}\right\} \ldots\ldots(92)$$

When $\alpha_{11} = \alpha_{1n}$, the determinant condition $(r_1 r_n - r_{1n}^2) \geq 0$, for $\gamma \geq 0$, goes over into $r_1(r_n - r_1) \geq 0$, for $\gamma \geq 0$.

We know that r_1 is ≥ 0, for $\gamma \geq 0$; thus $(r_n - r_1)$ is also ≥ 0, for $\gamma \geq 0$.

Further, the reality clause is fulfilled, and it is clear that both Y_1 and Y_2 of (92) are "positive real" functions.

From this we see that:

If a non-singular "positive real" α-matrix is such[113] *that α_{11} is $= \alpha_{1n}$, then this particular matrix satisfies both necessary and suffi-*

---

cause then we need to consider the positiveness of the real part and the reality clause only. The numerical Examples, however, are reduced to a consideration of the boundary only.

[112] P. 48.

[113] See *Example 7* (Appendix, p. 204).

cient conditions for having a network representation in the L-structure shown in Fig. 24.[114]

Note: The General Circuit Parameters for this structure are:

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} 1+\dfrac{Y_2}{Y_1} & \dfrac{1}{Y_1} \\ Y_2 & 1 \end{Vmatrix}$$

(**b**). *Let us now consider the L-structure shown in Fig. 25.*[114]

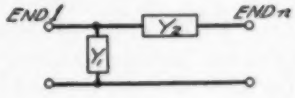


FIG. 25

For short-circuit at end 1 it is immediately clear that the structure corresponds to the case $\alpha_{nn} = \alpha_{1n}$.

Thus we get:

$$\left.\begin{aligned} \alpha_{11} &= Y_1 + Y_2 \\ \alpha_{nn} &= \alpha_{1n} = Y_2 \end{aligned}\right\} \qquad \ldots\ldots\ldots(93)$$

and when solving for the Y:s:

$$\left.\begin{aligned} Y_1 &= \alpha_{11} - \alpha_{nn} = (r_1 - r_n) + j(x_1 - x_n) \\ Y_2 &= \alpha_{nn} \quad\;\; = \quad r_n \quad\;\; + jx_n \end{aligned}\right\} \ldots\ldots(94)$$

Again we see that Y_1 and Y_2 are "positive real" functions, because when α_{nn} is $= \alpha_{1n}$, then the r-determinant condition goes over into: $r_n (r_1 - r_n) \geq 0$ for $\gamma \geq 0$.

We know that r_n is ≥ 0 for $\gamma \geq 0$. Thus $(r_1 - r_n)$ is also ≥ 0, for $\gamma \geq 0$.

Consequently:

If a non-singular "positive real" α-matrix is such[115] *that α_{nn} is $= \alpha_{1n}$, then this particular matrix satisfies both necessary and suffi-*

[114] Compare Bibliogr. 23; dealing with purely reactive structures.
[115] See *Example 8* (Appendix, p. 206).

cient conditions for having a network representation in the L-structure shown in Fig. 25.[116]

Note: The General Circuit Parameters for this structure are:

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} 1 & \frac{1}{Y_2} \\ Y_1 & 1+\frac{Y_1}{Y_2} \end{Vmatrix}$$

We notice, that in item (a) D is $= 1$, and in item (b) A is $= 1$. Further, in both Fig. 24 and in Fig. 25 the admittance of the vertical arm is nothing but the General Circuit Parameter C, and the reciprocal of the admittance of the horizontal arm nothing but the General Circuit Parameter B.

(c). *Let us now suppose that the General Circuit Parameter C is zero.* Then we get the structure shown in Fig. 26.[117]

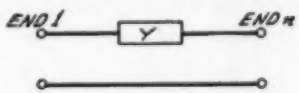


FIG. 26

It is immediately evident that this structure corresponds to the case when all three α:s are equal, i.e. we have

$$\alpha_{11} = \alpha_{nn} = \alpha_{1n} = \alpha' \ (say) = Y \quad \ldots\ldots (95)$$

Evidently?

If a "positive real" α-matrix is such[118] *that all its elements are equal, then this particular matrix satisfies both necessary and suffi-*

[116] Note, in case β:s and not α:s are prescribed, then if $\beta_{11} = \beta_{1n}$ the β-matrix would have a network representation in the L-structure shown in Fig. 25 and if $\beta_{nn} = \beta_{1n}$ the network representation would be the L-structure in Fig. 24.

[117] This does not mean that the structure in Fig. 26 takes care of all families of functions when $C = 0$, but only when in addition either one of the driving-point functions is equal to the transfer function.

[118] See *Example 9* (Appendix, p. 206).

cient conditions for having a network representation in the structure shown in Fig. 26.

Note: The General Circuit Parameters are:

$$\left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| = \left\| \begin{matrix} 1 & \frac{1}{Y} \\ 0 & 1 \end{matrix} \right\|$$

Here it should be pointed out that *the structure in Fig. 26 is not characterized by β-matrix*, because all β:s obtained by a transformation according to (16)[119] become infinite.

They also are infinite from a physical consideration.

Thus we say that the structure in Fig. 26 "*belongs*" to the α:s.

(**d**). *Let us now suppose that the General Circuit Parameter B is zero* in which case we get the structure shown in Fig. 27.[120]

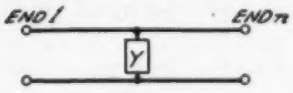


FIG. 27

It is evident that this structure cannot be characterized by an α-matrix, because all α:s are infinite.

On the other hand, it is completely specified by a β-matrix and corresponds to the case when all three β:s are equal, i.e. we have:

$$\beta_{11} = \beta_{22} = \beta_{12} = \beta' \text{ (say)} = \frac{1}{Y} \qquad \ldots\ldots (96)$$

Evidently:

If a "positive real" β-matrix is such[121] *that all its elements are equal, then this particular matrix satisfies both necessary and suffi-*

[119] P. 31; note in this case the α:s do not form a non-singular but a singular matrix, and consequently the β:s cannot be obtained from the α:s, because the two matrices are not each others inverses in this particular case.

[120] Same remark is due here as given on page 94 as to the case when C is zero.

[121] See *Example 10* (Appendix, p. 206).

cient conditions for having a network representation in the structure shown in Fig. 27.

Note: The General Circuit Parameters are:

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ Y & 1 \end{Vmatrix}$$

This structure "*belongs*" to the β:s.

(e). *Let us consider the T- (Z-) structure, shown in Fig. 28.*

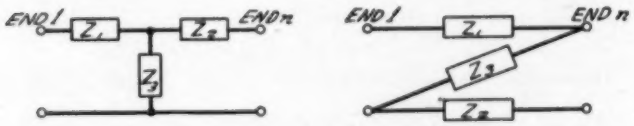


FIG. 28

These two structures are practically the same and have the same characteristics. They both "*belong*" to the β:s as will be shown in the following, and for either one of them we get:

$$\left.\begin{aligned} \beta_{11} &= Z_1 + Z_3 \\ \beta_{nn} &= Z_2 + Z_3 \\ \beta_{1n} &= Z_3 \end{aligned}\right\} \qquad \ldots\ldots(97)$$

i.e. when solving for the Z:s we get by the aid of (62):[122]

$$\left.\begin{aligned} Z_1 &= \beta_{11} - \beta_{1n} = (h_1 - h_{1n}) + j\,(v_1 - v_{1n}) \\ Z_2 &= \beta_{nn} - \beta_{1n} = (h_n - h_{1n}) + j\,(v_n - v_{1n}) \\ Z_3 &= \beta_{1n} \qquad\quad = h_{1n} \qquad + j\ v_{1n} \end{aligned}\right\} \ldots.(98)$$

From (98) we see that:

If a non-singular "positive real" β-matrix is such that the differences between the two driving-point functions and the transfer function are "positive real" functions, and if in addition β_{1n} is a "positive real" function itself, then this particular matrix satisfies

[122] P. 56.

both necessary and sufficient conditions for having a network representation in a T- (Z-) structure.

The General Circuit Parameters have in this case no specific feature, but are given by (22).[123]

Suppose we do not know to which family of functions a T-structure "belongs" and start out from the α:s instead of the β:s. Then exchanging the Z:s in Fig. 28 against Y:s, having corresponding subscripts, we get:

$$\left.\begin{aligned} \alpha_{11} &= \frac{Y_1(Y_2+Y_3)}{Y_1+Y_2+Y_3} = \frac{Y_1Y_2+Y_1Y_3}{Y_1+Y_2+Y_3} \\ \alpha_{nn} &= \frac{Y_2(Y_1+Y_3)}{Y_1+Y_2+Y_3} = \frac{Y_1Y_2+Y_2Y_3}{Y_1+Y_2+Y_3} \\ \alpha_{1n} &= \frac{Y_2}{Y_2+Y_3}\alpha_{11} = \frac{Y_1Y_2}{Y_1+Y_2+Y_3} \end{aligned}\right\} \quad \ldots\ldots(99)$$

i.e. when solving for the Y:s.

$$\left.\begin{aligned} Y_1 &= \frac{\alpha_{11}\alpha_{nn}-\alpha_{1n}^2}{\alpha_{nn}-\alpha_{1n}} \\ Y_2 &= \frac{\alpha_{11}\alpha_{nn}-\alpha_{1n}^2}{\alpha_{11}-\alpha_{1n}} \\ Y_3 &= \frac{\alpha_{11}\alpha_{nn}-\alpha_{1n}^2}{\alpha_{1n}} \end{aligned}\right\} \quad \ldots\ldots(100)$$

When now comparing (100) with (98), it is evident that the equations (98) have a distinct advantage over the other set as to clearness. Thus, when dealing with the α:s, it is in this case practically impossible to see whether each one of the Y:s of a T-structure is a "positive real" function or not, because the real part of each Y contains both r:s and x:s as shown on following page.

[123] P. 34.

$$
\left.
\begin{aligned}
ReY_1 &= \frac{\begin{matrix}[(r_1r_n - r_{1n}^2)\,r_n + r_nx_{1n}^2 + r_1x_n - 2x_nr_{1n}x_{1n}] - \\ -\,[(r_1r_n - r_{1n}^2)\,r_{1n} - r_{1n}x_1x_n + r_1x_nx_{1n} + r_nx_1x_{1n} + r_{1n}x_{1n}^2]\end{matrix}}{(r_n - r_{1n})^2 + (x_n - x_{1n})^2} \\
ReY_2 &= \frac{\begin{matrix}[(r_1r_n - r_{1n}^2)\,r_1 + r_1x_{1n}^2 + r_nx_1^2 - 2x_1r_{1n}x_{1n}] - \\ -\,[(r_1r_n - r_{1n}^2)\,r_{1n} - r_{1n}x_1x_n + r_nx_1x_{1n} + r_1x_nx_{1n} + r_{1n}x_{1n}^2]\end{matrix}}{(r_1 - r_{1n})^2 + (x_1 - x_{1n})^2} \\
ReY_3 &= \frac{[(r_1r_n - r_{1n}^2)\,r_{1n} - r_{1n}x_1x_n + r_nx_1x_{1n} + r_1x_nx_{1n} - r_{1n}x_{1n}^2]}{r_{1n}^2 + x_{1n}^2}
\end{aligned}
\right\} \ldots (101)^{124}
$$

From the curves of *Example 11*[125] it is interesting to note the fact that when dealing with the α:s their real and imaginary parts behave in a very peculiar manner, while the real and imaginary parts of corresponding β:s are very regular, and the T-structure, being the network representation of the prescribed functions, is the most simple one.

(f). *Let us now consider the π-structure, shown in Fig. 29.*

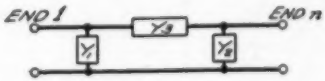


FIG. 29

Contrary to the T-structure it "*belongs*" to the α:s, and we get:

$$
\left.
\begin{aligned}
\alpha_{11} &= Y_1 + Y_3 \\
\alpha_{nn} &= Y_2 + Y_3 \\
\alpha_{1n} &= Y_3
\end{aligned}
\right\} \qquad \ldots\ldots (102)
$$

i.e. when solving for the Y:s:

$$
\left.
\begin{aligned}
Y_1 &= \alpha_{11} - \alpha_{1n} = (r_1 - r_{1n}) + j\,(x_1 - x_{1n}) \\
Y_2 &= \alpha_{nn} - \alpha_{1n} = (r_n - r_{1n}) + j\,(x_n - x_{1n}) \\
Y_3 &= \alpha_{1n} \qquad\quad = \quad r_{1n} \quad + jx_{1n}
\end{aligned}
\right\} \ldots (103)
$$

[124] Note the lower line in the numerators of $\mathrm{Re}Y_1$ and $\mathrm{Re}Y_2$ is a continuation of the upper line.

[125] Appendix, p. 206.

Equations (103) clearly tell us that:

If a non-singular "positive real" α-matrix is such that the differences between the driving-point functions and the transfer function are "positive real" functions, and if in addition α_{1n} is a "positive real" function itself, then this particular matrix satisfies both necessary and sufficient conditions for having a network representation in a π-structure.

The General Circuit Parameters have here no specific feature.

If instead we had chosen to work with the β:s, we would have obtained expressions exactly analogous to those of (99), (100) and (101) when considering the α:s in connection with a T-structure, and the same discussion as given there would hold here.

Example 12,[126] considered together with *Example* 11,[126] clearly brings out the analogy between the two structures.

Except in the case, when the elements of the α- or β-matrix are in a constant ratio a T-structure has for all real ω:s no equivalent π-structure and vice versa. In the Examples mentioned the equivalence holds for one single frequency only,[127] i.e. for $\omega = \frac{1}{10}$.

(g). *The bridged T-structure*, shown in Fig. 30, is of minor importance.

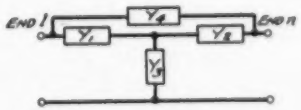


FIG. 30

When solving for the α:s say, in terms of the Y:s we get three equations but four unknowns, and thus it is clear that the Y:s cannot be expressed in the α:s alone but three of the Y:s can only be expressed in the α:s together with the fourth Y.

[126] Appendix, p. 208.

[127] Actually $\omega = 0$ also will do, but for this frequency the whole π-structure (T-structure respectively) reduces to a non-reactive two-terminal network.

(h). *In this item we are going to consider the two lattice-structures, the symmetrical and the unsymmetrical*, and start with the first one shown in Fig. 31.[128]

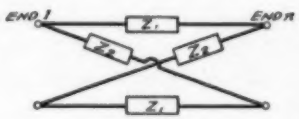


FIG. 31

(1) For this structure the two driving-point functions are evidently equal, i.e. it is characterized by *two* functions only.

Under next sub-item the three α:s, say, for the unsymmetrical lattice structure will be written down by inspection. Here, however, let us adopt a somewhat different method.

From Fig. 31 it is clear that the short circuit (Z_k) and open-circuit (Z_o) impedances for either end evidently are:

$$\left.\begin{aligned} Z_k &= \frac{2Z_1Z_2}{Z_1 + Z_2} \\ Z_0 &= \frac{Z_1 + Z_2}{2} \end{aligned}\right\}$$

i.e. we get:

$$\left.\begin{aligned} Z_0Z_k &= Z_1Z_2 \\ 2Z_0 &= Z_1 + Z_2 \end{aligned}\right\}$$

Knowing the product and the sum of the roots, we get the second degree equation:

$$x^2 - 2Z_0x + Z_0Z_k = 0$$

i.e.

$$x = Z_0 \pm \sqrt{Z_0^2 - Z_0Z_k}$$

[128] Compare Bibliogr. 23, dealing with the purely reactive case.

From equations (19)[129], however, we see that Z_a is $= \frac{A}{C}$ and Z_b is $= \frac{B}{D}$. Substituting these values into the above expression, we get:

$$x = \frac{A}{C} \pm \sqrt{\frac{A^2}{C^2} - \frac{AB}{CD}} = \frac{A}{C} \pm \sqrt{\frac{A\,(AD - BC)}{C^2 D}}$$

But $(AD - BC)$ is $= 1$, and A is $= D$.
Consequently:

$$\left.\begin{aligned} x &= \frac{A \pm 1}{C} \\ Z_1 &= \frac{A+1}{C}\left(\text{or } \frac{A-1}{C}\right) \\ Z_2 &= \frac{A-1}{C}\left(\text{or } \frac{A+1}{C}\right) \end{aligned}\right\} \quad \ldots\ldots(104)$$

Note from (104) the fact that Z_1 *and* Z_2 *are interchangeable.*

Then, by the aid of equations (22)[129] the above expressions for Z_1 and Z_2 go over into:

$$\left.\begin{aligned} Z_1 &= \beta_{11} + \beta_{1a} \\ Z_2 &= \beta_{11} - \beta_{1a} \end{aligned}\right\} \quad \ldots\ldots(105)$$

or when substituting for A and C, in terms of the α:s, into (104), we get:

$$\left.\begin{aligned} Z_1 &= \frac{1}{\alpha_{11} - \alpha_{1a}} \\ Z_2 &= \frac{1}{\alpha_{11} + \alpha_{1a}} \end{aligned}\right\}$$

[129] P. 33.

Thus, if letting the two-terminal networks of Fig. 31 be denoted as admittances instead of impedances (the subscripts remaining the same) we have:

$$\left.\begin{aligned} Y_1 &= \alpha_{11} - \alpha_{1n} \\ Y_2 &= \alpha_{11} + \alpha_{1n} \end{aligned}\right\} \quad \ldots\ldots(106)$$

Then from equations (106) say, and (47) we get:

$$\left.\begin{aligned} Y_1 &= (r_1 - r_{1n}) + j\,(x_1 - x_{1n}) \\ Y_2 &= (r_1 + r_{1n}) + j\,(x_1 + x_{1n}) \end{aligned}\right\} \quad \ldots\ldots(107)$$

From (107) we see that the two Y:s are "positive real" functions, because for $\alpha_{11} = \alpha_{nn}$ the r-determinant condition: $(r_1 r_n - r_{1n}^2) \geq 0$, goes over to: $(r_1^2 - r_{1n}^2) \geq 0$, i.e. to: $(r_1 + r_{1n})(r_1 - r_{1n}) \geq 0$. Thus, as the r-determinant condition always is satisfied, then $(r_1 + r_{1n})$ and $(r_1 - r_{1n})$ of (107) *are* individually ≥ 0, and we can state:

If a non-singular "positive real" α- or β-matrix is such that the two elements located on its principal diagonal are equal, then this particular matrix satisfies both necessary and sufficient conditions for having a network representation in a symmetrical[130] *lattice-structure.*

Note, the General Circuit Parameters A and D are here equal.

(2) *We now proceed to consider the unsymmetrical lattice-structure* of Fig. 32, and let us first deal with the α:s (say) considered as "currents."

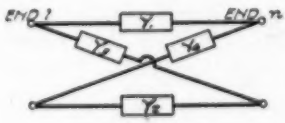


FIG. 32

[130] Note the fact that for the driving-point functions to be equal the corresponding lattice-structure need not be symmetrical as illustrated in *Example* 13 (Appendix, p. 209). An equivalent symmetrical lattice-structure always exists, however, as shown in the same Example. As to earlier discussions of the symmetrical lattice, structure and the L-structures see p. 25 and Bibliogr. 31, 22 and 23, etc.

α_{11} and α_{nn} can immediately be written down by inspection only, but as to α_{1n} it cannot be found in exactly the same simple manner.

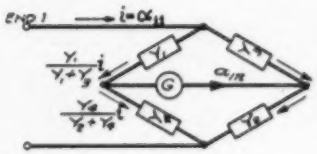


FIG. 33

From Fig. 33, however, we get:

$$\alpha_{1n} = \frac{Y_1}{Y_1 + Y_3} i - \frac{Y_4}{Y_2 + Y_4} i = \frac{Y_3}{Y_1 + Y_3} i - \frac{Y_2}{Y_2 + Y_4} i$$

Note "i" is nothing but α_{11}, and thus we have our three α:s:

$$\left.\begin{aligned} \alpha_{11} &= \frac{(Y_1 + Y_3)(Y_2 + Y_4)}{Y_1 + Y_2 + Y_3 + Y_4} = \frac{Y_1Y_2 + Y_1Y_4 + Y_2Y_3 + Y_3Y_4}{Y_1 + Y_2 + Y_3 + Y_4} \\ \alpha_{nn} &= \frac{(Y_1 + Y_4)(Y_2 + Y_3)}{Y_1 + Y_2 + Y_3 + Y_4} = \frac{Y_1Y_2 + Y_1Y_3 + Y_2Y_4 + Y_3Y_4}{Y_1 + Y_2 + Y_3 + Y_4} \\ \alpha_{1n} &= \left[\frac{Y_1}{Y_1 + Y_3} - \frac{Y_4}{Y_2 + Y_4}\right]\alpha_{11} = \frac{Y_1Y_2 - Y_3Y_4}{Y_1 + Y_2 + Y_3 + Y_4} \end{aligned}\right\} \quad . . (108)$$

Equations (108), however, contain four unknown and thus they cannot be solved generally, i.e. each Y cannot individually be expressed in the α:s.

The General Solution of (108) is found[131] to be:

$$\left.\begin{aligned} Y_3 &= \frac{MN - a\alpha_{1n}}{a} \\ Y_1 &= M - Y_3 \quad = \frac{a(M + \alpha_{1n}) - MN}{a} \\ Y_2 &= N - Y_3 \quad = \frac{a(N + \alpha_{1n}) - MN}{a} \\ Y_4 &= a - (N + Y_1) = \frac{a^2 - a(M + N + \alpha_{1n}) + MN}{a} \end{aligned}\right\} \quad \ldots(109)$$

where:

$$\left.\begin{aligned} a &= Y_1 + Y_2 + Y_3 + Y_4 \\ M &= \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4\,a\,\alpha_{11}} \\ N &= \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4\,a\,\alpha_{nn}} \end{aligned}\right\}$$

i.e. a, M and N are all functions of λ.

What bothers us here is the square roots. These, however, can be dispensed with in the following manner:[132]

Let us introduce two new quantities and at the same time put:

$$\left.\begin{aligned} a &= \frac{4\,\alpha_{11}}{1 - \zeta^2} \\ a &= \frac{4\,\alpha_{nn}}{1 - \mu^2} \end{aligned}\right\} \quad \ldots\ldots(110)$$

Thus:

$$\frac{\alpha_{11}}{(1 + \zeta)(1 - \zeta)} = \frac{\alpha_{nn}}{(1 + \mu)(1 - \mu)} \quad \ldots\ldots(111)$$

[131] Mainly due to Dr. *D. T. Struik* of the Mathematical Department of The Massachusetts Institute of Technology.

[132] This method is very old (used by mathematicians several hundred years ago) and was suggested by Dr. *P. Franklin* of the Mathematical Department of the Massachusetts Institute of Technology.

Further let:

$$\frac{\alpha_{11}(1-\mu)}{1-\zeta} = \frac{\alpha_{nn}(1+\zeta)}{1+\mu} be = \nu \qquad \ldots\ldots(112)$$

where ν corresponds to a short-circuit driving point admittance for $\alpha_{11} = \alpha_{nn}$.[133]

From (112) we get the two equations:

$$\left.\begin{aligned} \frac{\alpha_{11}}{1-\zeta} &= \frac{\nu}{1-\mu} \\ \frac{\alpha_{nn}}{1+\mu} &= \frac{\nu}{1+\zeta} \end{aligned}\right\}$$

the solution of which gives:

$$\left.\begin{aligned} \zeta &= \frac{\nu^2 + \alpha_{11}\alpha_{nn} - 2\nu\,\alpha_{11}}{\nu^2 - \alpha_{11}\alpha_{nn}} \\ \mu &= \frac{-\nu^2 + 2\nu\,\alpha_{nn} - \alpha_{11}\alpha_{nn}}{\nu^2 - \alpha_{11}\alpha_{nn}} \end{aligned}\right\}$$

When substituting above ζ and μ into any one of the two expressions of (110) we obtain the general form of "a" as:

$$a = \frac{(\nu^2 - \alpha_{11}\alpha_{nn})^2}{\nu(\nu - \alpha_{11})(\nu - \alpha_{nn})} \qquad \ldots\ldots(113)$$

and when then substituting this expression for "a" into M and N, they become:

(1) for plus signs (in front of the radical):

$$\left.\begin{aligned} M_1 &= \frac{\nu^2 - \alpha_{11}\alpha_{nn}}{\nu - \alpha_{nn}} \\ N_1 &= \frac{\alpha_{nn}(\nu^2 - \alpha_{11}\alpha_{nn})}{\nu(\nu - \alpha_{nn})} \end{aligned}\right\}$$

[133] Note ζ and μ are no constants but functions of λ, and ν is a similar function. Eventually ζ and μ are eliminated, and ν is the arbitrary function replacing "a" in (109).

(2) For minus signs:

$$\left.\begin{aligned} M_2 &= \frac{\alpha_{11}(\nu^2 - \alpha_{11}\alpha_{nn})}{\nu(\nu - \alpha_{11})} \\ N_2 &= \frac{\nu^2 - \alpha_{11}\alpha_{nn}}{\nu - \alpha_{11}} \end{aligned}\right\}$$

Using M_1 and N_1 (say) we get:

$$\left.\begin{aligned} Y_3 &= \alpha_{nn}\left[\frac{\nu - \alpha_{11}}{\nu - \alpha_{nn}}\right] - \alpha_{1n} \\ Y_1 &= \nu + \alpha_{1n} \\ Y_2 &= \frac{\alpha_{11}\alpha_{nn}}{\nu} + \alpha_{1n} \\ Y_4 &= \alpha_{11}\left[\frac{\nu - \alpha_{nn}}{\nu - \alpha_{11}}\right] - \alpha_{1n} \end{aligned}\right\} \quad \ldots\ldots(114)$$

On the other hand if using M_2 and N_2 we get a similar set of equations, except Y_2 and Y_4 as well as Y_1 and Y_2 are interchanged.

Note above "solution" is still general which can be seen from the fact that when letting $\alpha_{11} = \alpha_{nn} = \nu$, then Y_1 becomes equal to Y_2 and at the same time Y_3 becomes equal to Y_4, i.e. we are back to the symmetrical lattice-structure dealt with under sub-item (1).

Instead of dealing with ν, let us substitute its value $(Y_1 - \alpha_{1n})$ into the other three Y:s, getting:

$$\left.\begin{aligned} Y_3 &= \alpha_{nn}\frac{Y_1 - (\alpha_{11} + \alpha_{nn})}{Y_1 - (\alpha_{nn} + \alpha_{1n})} - \alpha_{1n} \\ Y_2 &= \frac{\alpha_{11}\alpha_{nn}}{Y_1 - \alpha_{1n}} + \alpha_{1n} \\ Y_4 &= \alpha_{11}\frac{Y_1 - (\alpha_{nn} + \alpha_{1n})}{Y_1 - (\alpha_{11} + \alpha_{1n})} - \alpha_{1n} \end{aligned}\right\} \quad \ldots\ldots(115)$$

When using M_2 and N_2 we naturally get analogous expressions to (115).

If we now let Y_1 be such a "positive real" function that when substituted into (115) it together with the α:s make each one of Y_2, Y_3 and Y_4 individually a "positive real" function, then this structure can be used as a network representation of the prescribed α:s under consideration.

It may be pointed out that when dealing with the lattice-structure we could equally well have considered the β:s and denoted the two-terminal networks as impedances (Z:s) instead of admittances (Y:s).

For a simple numerical demonstration of the structure dealt with in this item see *Example* 2.[134]

(i). During the progress of our research *it soon became clear to the writer, that in many cases the network representation of a family of α:s or β:s* MUST *include a transformer.* So, for instance, if the prescribed α:s are constants and such that α_{1n} lies between α_{11} and α_{nn} in magnitude, which is allowed although their determinant must be ≥ 0, then the necessity of a transformer is self-evident, for in no other way can a current in the short-circuit at the distant end be greater than the current[135] flowing in at the end where the e.m.f. is applied.

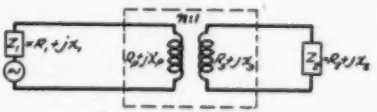


FIG. 34

The structure shown inside the dotted lines in Fig. 34 is an ideal[136] transformer having a ratio of turns n:1.

[134] Appendix, p. 200.

[135] Note, both currents are in phase.

[136] In spite of the fact that such a transformer scarcely can be built, it is nevertheless mathematically usable and is extensively used in the literature on network synthesis. Without hesitation we talk about "pure reactances," etc., but on the

Let us now for a moment discuss this structure in connection with our α:s and β:s.

To our "surprise" *we will find that neither set of them characterizes an ideal transformer*, but note it is always used together with some other structure.

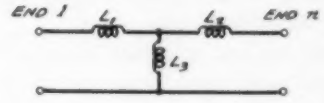


FIG. 35

Representing the transformer as shown in Fig. 35 and writing down our α:s by inspection, we get:

$$\left.\begin{aligned}
\alpha_{11} &= \frac{\dfrac{1}{L_1\lambda}\left[\dfrac{1}{L_2\lambda}+\dfrac{1}{L_3\lambda}\right]}{\dfrac{1}{L_1\lambda}+\dfrac{1}{L_2\lambda}+\dfrac{1}{L_3\lambda}} = \frac{L_2+L_3}{\lambda(L_1L_2+L_1L_3+L_2L_3)} \\
\alpha_{nn} &= \frac{\dfrac{1}{L_2\lambda}\left[\dfrac{1}{L_1\lambda}+\dfrac{1}{L_3\lambda}\right]}{\dfrac{1}{L_1\lambda}+\dfrac{1}{L_2\lambda}+\dfrac{1}{L_3\lambda}} = \frac{L_1+L_3}{\lambda(L_1L_2+L_1L_3+L_2L_3)} \\
\alpha_{1n} &= \frac{\dfrac{1}{L_2\lambda}}{\dfrac{1}{L_2\lambda}+\dfrac{1}{L_3\lambda}}\,\alpha_{11} = \frac{L_3}{\lambda(L_1L_2+L_1L_3+L_2L_3)}
\end{aligned}\right\}. \quad (116)$$

But note, the sum $(L_1L_2 + L_1L_3 + L_2L_4)$, corresponding to close coupling[137] between two inductances (i.e. the case we have

other hand we know that an inductance coil always contains a small amount of resistance and some distributed capacitance and that a condenser always has some leakage between its plates and same resistance in its connecting leads. At frequencies above the quasi-stationary limit the reactance of the leads becomes important.

[137] See Bibliogr. 38, p. 49.

to do with here), is = 0 for all ω:s, and thus from (116) we see that all α:s become infinite.

As to the β:s they are for an ideal transformer practically inconceivable. For zero frequency they are all zero, but are increasing towards infinity as the frequency increases.

The General Circuit Parameters, however, specify the structure completely, and we shall now see what they are in this case.

It is immediately clear that for an ideal transformer the General Circuit Parameter C[138] is zero, due to the fact that we have no magnetizing current, i.e. the primary ampere turns are always equal to the secondary.

Further, it is easy to see that B[138] is also zero, because we have no leakage and nothing is connected in series with the windings.

Thus the relation $(AD - BC) = 1$ reduces to $AD = 1$, and from (19), when considering the relationships between number of turns and voltages or currents respectively, we get:

$$\left.\begin{aligned} A &= \frac{E_1}{E_n} = n{:}1 \\ D &= \frac{I_1}{I_n} = 1{:}n \end{aligned}\right\}$$

i.e. A and D are for an ideal transfer each others inverses, A being the ratio of turns between primary[139] and secondary, and D ditto between secondary and primary.

[138] Note, in practice where we will have to use non-ideal transformers, we must remember that the consumed magnetizing current corresponds to an impedance placed in parallel with either one of the windings, and the leakage corresponds to an impedance placed in series with either one of the windings. As to the reactance character of above impedances it depends upon where on the frequency scale the network is going to be used. For lower frequencies the inductive feature, for higher the capacitive feature, should predominate.

[139] "*Primary*" refers to "*end 1*," and "*secondary*" to "*end n*."

Consequently the matrix, characterizing an ideal transformer, is:

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} A & 0 \\ 0 & D \end{Vmatrix} = \begin{Vmatrix} \frac{1}{D} & 0 \\ 0 & D \end{Vmatrix} = \begin{Vmatrix} A & 0 \\ 0 & \frac{1}{A} \end{Vmatrix} = \begin{Vmatrix} n & 0 \\ 0 & \frac{1}{n} \end{Vmatrix}$$

The use of the transformer together with another structure will be taken up in the next Part.

PART VII

A GENERAL SURVEY OF MATRIX- AND DETERMINANT FEATURES, AND THE APPLICATION OF THE ALGEBRAIC LAWS; THE REALIZATION OF CERTAIN TYPES OF "POSITIVE REAL" MATRICES, ETC.

For a clear understanding of the following a few features embodied in matrices in general compared with those of corresponding determinants will be given. Then we proceed to consider determinants and matrices in respect to the algebraic laws and the physical interpretation of certain "positive real" matrices etc.

(**a**). There is no such thing as a determinant having different numbers of rows and columns, while the number of rows and columns of a matrix need not neeessarily be the same.

Two determinants may be equal even if their corresponding elements are different, but two matrices are equal when and only when, they have the same number of rows and columns, and in addition every element of one matrix must be equal to the corresponding element of the other.

The elements of a determinant may all be finite and still the *value* of the determinant may be zero, while the value of a matrix cannot be given by a single number except by special conventions, i.e. "zero matrix," "unit matrix," "scalar matrix", etc.

The sum or difference of two determinants is purely algebraic, while with the sum or difference of two matrices, having m rows and n columns each, we generally understand a matrix, which also has m rows and n columns, each of whose elements is the sum or difference respectively of the corresponding elements of the given matrices.

The ordinary algebraic laws hold for all operations with determinants, while for matrices, they hold only for addition or subtraction and for their multiplication by scalars.

In our problem we only meet matrices having the same number of rows and columns, i.e. *square matrices*, and we shall now consider the cases of multiplication and division of such matrices.

(**b**). The product of two determinants of the same order may be written in four different ways and the multiplication procedure can equally well be performed in six different ways, i.e. either by multiplying each element of each row of one determinant with each element of all columns of the other, or vice versa, and by multiplying column(s) with column(s) or row(s) with row(s).

When dealing with matrices the above arbitrariness no longer applies.

The place of their elements is fixed from the beginning, and the multiplication procedure can be performed in only one way, which is by multiplying each row of the left matrix in sequence with all columns of the right matrix, i.e. for instance:

$$\begin{Vmatrix} A' & B' \\ C' & D' \end{Vmatrix} * \begin{Vmatrix} A'' & B'' \\ C'' & D'' \end{Vmatrix} = \begin{Vmatrix} A'A'' + B'C'' & A'B'' + B'D'' \\ C'A'' + D'C'' & D'B'' + D'D'' \end{Vmatrix}. \quad (117)$$

The matrix equation (117) embodies a very important *physical* fact, because assuming the letters used are the General Circuit Parameters of two separate four-terminal networks as shown below, and we want to find the corresponding quantities for the combination after end n of the left network is connected to end 1 of the right one.

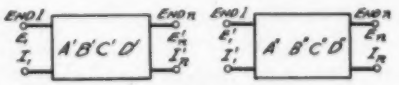


FIG. 36

The usual way of obtaining these Parameters, as often shown in the literature,[140] is to set up equations (19)[141] for each network separately and then eliminate E'_n, I'_n, E'_1 and I'_1.

In doing so, however, we will find that the coefficients for I_n and E_n in the result *are nothing but the elements of the matrix to the right of the equality sign in* (117).

In the same way if several four-terminal networks, characterized by the matrices $M_1, M_2, M_3, M_4, \ldots M_n$, are connected together in a manner as indicated in Fig. 36, in which case the substitution method becomes very laborious if not prohibitive, the resulting General Circuit Parameters of the combination, being elements of a matrix M_0, say, is found from:

$$M_0 = M_1 \times M_2 \times M_3 \times M_4 \times \ldots \times M_n \qquad ..(118)$$

i.e. all voltages and currents, except those at the input and output terminals of the combination, so to say vanish from the picture, and all that we need to do is to perform a simple matrix multiplication.[142]

As to matrix multiplication not all the laws of ordinary algebra hold.[143]

The multiplication of three matrices (a, b and c say) *is associative*, i.e.

$$(ab)c = a(bc)$$

Further, it *is distributive*, i.e.

$$a(b + c) = ab + ac$$

It is not commutative, however, i.e. in general:

$$ab \neq ba$$

This latter feature introduces two kinds of matrix division, as will be shown under item (d).

[140] See, for instance, Bibliogr. 10, p. 251 (there, however, the Parameters are Constants).
[141] P. 33.
[142] Bibliogr. 25, p. 102.
[143] Bibliogr. 49, p. 63.

(c). When considering a single frequency only any "positive real" α- or β-matrix can readily be realized by first translating the prescribed matrix into another matrix[144] whose elements are the General Circuit Parameters, which in this case are constants, and then split up this matrix into a product of component matrices each of which being realizable into one of the standard structures dealt with under the previous Part, whereupon the separate structures are joined together in chain connection to a total network representation.

To solve the general problem by this procedure is evidently hopeless, however, because even if the original matrix fulfils all requirements of having its correspondence in a "positive real" α- or β-matrix, its component matrices may or may not have this feature.

On the contrary if in a product of several "complex,"[145] matrices, having as elements the General Circuit Parameters, each matrix has its correspondence in a "positive real" matrix, then the matrix representing the product has also its correspondence in a "positive real" matrix, as briefly outlined below.

Let the elements of the resulting matrix of (117) be the four General Circuit Parameters A, B, C and D. Further, let the three α-families (say), corresponding to (117), be denoted in an analogous manner to the three sets of General Circuit Parameters.

Then we get, by the aid of (22):

$$\left.\begin{aligned} \alpha_{11} &= \frac{D}{B} = \frac{C'B'' + D'D''}{A'B'' + B'D''} \\ \alpha_{nn} &= \frac{A}{B} = \frac{A'A'' + B'C''}{A'B'' + B'D''} \\ \alpha_{1n} &= \frac{1}{B} = \frac{1}{A'B'' + B'D''} \end{aligned}\right\} \quad \ldots\ldots (119)$$

[144] Note, this matrix is according to our definition on pp. 85–88 no "positive real" matrix.

[145] With this we mean a matrix whose elements are complex quantities.

Again using (22) in the reverse sense to the above, then (119) goes over into:

$$\left.\begin{aligned} \alpha_{11} &= \alpha'_{11} - \frac{\alpha'^{2}_{1n}}{\alpha'_{nn} + \alpha''_{11}} \\ \alpha_{nn} &= \alpha''_{nn} - \frac{\alpha''^{2}_{1n}}{\alpha'_{nn} + \alpha''_{11}} \\ \alpha_{1n} &= \frac{\alpha'_{1n}\alpha''_{1n}}{\alpha'_{nn} + \alpha''_{11}} \end{aligned}\right\}$$

Then considering the whole right half of the λ-plane and splitting up all α:s into their real and imaginary parts (denoted in a corresponding manner as the α:s themselves) according to (47), we get:

$$\left.\begin{aligned} r_1 &= r'_1 - \frac{(r'^{2}_{1n} - x'^{2}_{1n})(r'_n + r''_1) + 2r'_{1n}x'_{1n}(x'_n + x''_1)}{(r'_n + r''_1)^2 + (x'_n + x''_1)^2} \\ r_n &= r''_n - \frac{(r''^{2}_{1n} - x''^{2}_{1n})(r'_n + r''_1) + 2r''_{1n}x''_{1n}(x'_n + x''_1)}{(- - -)^2 + (- - -)^2} \\ r_{1n} &= \frac{(r'_{1n}r''_{1n} - x'_{1n}x''_{1n})(r'_n + r''_1) + (r'_{1n}x''_{1n} + x'_{1n}r''_{1n})(x'_n + x''_1)}{(- - -)^2 + (- - -)^2} \end{aligned}\right\} \quad \ldots (120)$$

We know that:

$$r'_1 \geq 0, r'_n \geq 0 \text{ and } (r'_1 r'_n - r'^{2}_{1n}) \geq 0, \text{ for } \gamma \geq 0$$

and

$$r''_1 \geq 0\ r''_n \geq 0 \text{ and } (r''_1 r''_n - r''^{2}_{1n}) \geq 0, \text{ for } \gamma \geq 0$$

To prove that necessarily also:

$$r_1 \geq 0, r_n \geq 0 \text{ and } (r_1 r_n - r^2_{1n}) \geq 0, \text{ for } \gamma \geq 0.$$

As being rather long and tedious the proof is not included in this paper, but is to be found in the author's dissertation and it is evident that above proof holds as well in terms of the β:s.

Further it is clear that it can again be repeated for the multiplication of the product matrix and another matrix etc.

The reality-clause is fulfilled, and thus we have material for another Theorem, the truth of which is self evident from the physical point of view.

THEOREM VI:

If in a product of two or several square matrices the elements of each component matrix are such that the matrix as a whole corresponds to a "positive real" α- or β-matrix, then the resulting matrix has also analogous correspondence in a "positive real" α- or β-matrix.

(**d**). *Below we shall take up the question of how to divide a matrix by another,*[146] the knowledge of which is fundamental for a clear understanding of the following four items.

From item (b) we see that multiplication of matrices is associative, and distributive, but not commutative.

Normally, we define division as the inverse of multiplication, but since matrix multiplication is not commutative, we get two distinct kinds of division.

In spite of this ambiguity, however, we find that if "a" is any matrix and "b" is any non-singular[147] matrix, there exists one, and only one, matrix X, which satisfies the equation:

$$a = bX$$

and one, and only one, matrix Y, which satisfies the equation:

$$a = Yb$$

Then the matrices X and Y are given by the formulas:

$$\left.\begin{aligned} X &= b^{-1}a \\ Y &= ab^{-1} \end{aligned}\right\} \qquad \ldots\ldots(121)$$

where b^{-1} stands for the inverse[148] matrix of b.

[146] Bibliogr. 49, p. 74.
[147] I.e., the determinant of its elements is zero.
[148] *Note, non-singular matrices only have inverses, while every matrix has an adjoint!*

Note, if "a" had been a scalar[149] matrix, then in (121) the product $b^{-1}a$ would have been $= ab^{-1}$, and consequently X would be $= Y$.

The "idemfactor" is such a scalar matrix.

In the following four items always let the matrices of (121) be as follows:

a: is the matrix containing the General Circuit Parameters corresponding to the three α:s or β:s to be realized.

b: is the transformer matrix:

$$\left\| \begin{matrix} n & o \\ o & \dfrac{1}{n} \end{matrix} \right\|$$

X and Y: are the component matrices, realizable into a certain kind of a standard structure.

Below we shall show how to realize a "positive real" α- or β-matrix, having the specific feature that a constant ratio exists between certain or all of its elements. The network representations of such matrices will play an important rôle as component networks in the total network representation of the total functions.

In the event that the prescribed functions happen initially to be elements of such a matrix, then they fall under the items below directly.

(e). Let us first consider the case *when the ratio between* α_{11} *and* α_{1n} (correspondingly, the ratio between β_{nn} and β_{1n}) *is a real constant* (positive or negative).

From (22) it is clear that this ratio is simply the General Circuit Parameter D, and when considering the α:s (say) we see that the prescribed functions are: α_{11}, α_{nn} and $\dfrac{\alpha_{11}}{D}$.

[149] I.e., a matrix in which all elements on its principal diagonal are the same while all other elements are zeros.

Suppose the prescribed functions fulfil the requirements of being elements of a "positive real" matrix, corresponding to a matrix whose elements are the General Circuit Parameters $A\ B\ C\ D$; this latter matrix can be written:

$$\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \begin{Vmatrix} \frac{1}{D} & 0 \\ 0 & D \end{Vmatrix} \times \begin{Vmatrix} DA & BD \\ \frac{C}{D} & 1 \end{Vmatrix} = \begin{Vmatrix} DA & \frac{B}{D} \\ CD & 1 \end{Vmatrix} \times \begin{Vmatrix} \frac{1}{D} & 0 \\ 0 & D \end{Vmatrix}. \quad (122)$$

←—Alternative I.—→ ←Alternative II.→

All component matrices of (122) we recognize immediately. The products evidently consist of a transformer matrix and a matrix specifying the L-structure shown in Fig. 24 and vice versa for the two alternatives.

The idea of writing down (122) as it stands, the writer got from (121) and note:

$\begin{Vmatrix} \frac{1}{D} & 0 \\ 0 & D \end{Vmatrix}$ is nothing but b, i.e.

$\begin{Vmatrix} D & 0 \\ 0 & \frac{1}{D} \end{Vmatrix}$ is nothing but b^{-1} (because the determinant of b is equal to 1).

Further:

$\begin{Vmatrix} DA & BD \\ \frac{C}{D} & 1 \end{Vmatrix}$, is nothing but X.

and

$\begin{Vmatrix} DA & \frac{B}{D} \\ CD & 1 \end{Vmatrix}$ is nothing but Y.

The given functions are: α_{11}, α_{nn} and $\frac{\alpha_{11}}{D}$, i.e. their real parts are: r_1, r_n and $\frac{r_1}{D}$.

As to the positiveness of the real parts, we know that:

$$\left.\begin{aligned} r_1 &\geq 0, \text{ for } \gamma \geq 0 \\ r_n &\geq 0, \text{ for } \gamma \geq 0 \\ \left(r_1 r_n - \frac{r_1^2}{D^2}\right) &= \frac{r_1}{D^2}(D^2 r_n - r_1) \geq 0, \text{ for } \gamma \geq 0 \end{aligned}\right\} \quad \ldots\ldots(123)$$

Alternative 1. Adding a prime to the α:s which correspond to the L-structure, they are by the aid of (22) and (122) found to be:

$$\left.\begin{aligned} \alpha'_{11} &= \alpha'_{1n} = \frac{1}{BD} = \frac{\alpha_{11}}{D^2} \\ \alpha'_{nn} &= \frac{AD}{BD} = \frac{A}{B} = \alpha_{nn} \end{aligned}\right\} \quad \ldots\ldots(124)$$

Consequently we see that $Re\alpha'_{11}$ and $Re\alpha'_{nn}$ are each ≥ 0, for $\gamma \geq 0$.

The determinant becomes:

$$\left(\frac{r_1}{D^2} r_n - \frac{r_1^2}{D^4}\right) = \frac{r_1}{D^4}(D^2 r_n - r_1)$$

which also is ≥ 0, for $\gamma \geq 0$, because each part of the determinant of (123) is ≥ 0, for $\gamma \geq 0$.

Thus, the L-structure is passive and we get the passive combination shown in Fig. 37.

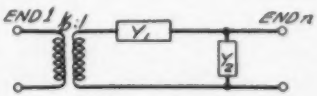


FIG. 37

The admittances of this structure (the Y:s) expressed in the General Circuit Parameters, by the aid of (92), (22), (122) and (124), are simply:

$$\left.\begin{aligned} Y_1 &= \frac{1}{BD} \\ Y_2 &= \frac{C}{D} \end{aligned}\right\} \qquad \ldots\ldots\ldots(125)$$

Alternative II. Letting the α:s, referring to the L-structure, in this case be α'':s we get:

$$\left.\begin{aligned} \alpha''_{11} &= \alpha''_{1n} = \frac{D}{B} = \alpha_{1n}D = \alpha_{11} \\ \alpha''_{nn} &= \frac{D^2A}{B} = D^2\alpha_{nn} \end{aligned}\right\} \qquad \ldots\ldots(126)$$

Thus $Re\alpha''_{11}$ and $Re\alpha''_{nn}$ are each ≥ 0, for $\gamma \geq 0$.

The determinant in this case becomes:

$$(r_1D^2r_n - r_1^2) = r_1(D^2r_n - r_1)$$

which also is ≥ 0 for $\gamma \geq 0$, due to (123).

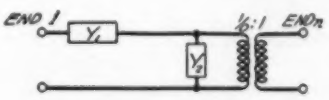


FIG. 38

Thus we get the passive structure shown in Fig. 38 and the Y:s are simply:

$$\left.\begin{aligned} Y_1 &= \frac{D}{B} \\ Y_2 &= CD \end{aligned}\right\} \qquad \ldots\ldots(127)$$

The reality clause is fulfilled, and due to above two alternatives we can state:

THEOREM VII:

If a non-singular "positive real" α- or β-matrix is such that the General Circuit Parameter D, to be obtained from this matrix, is a finite, real (positive or negative; if negative we only transpose one pair of terminals, or wind the transformer as to take care of the negative sign) *constant, then this particular matrix satisfies both necessary and sufficient conditions for having a network representation in either one of the structures shown in Fig. 37 or 38*[150] (if above ratio is = 1, then the transformer may or may not be necessary).

For an illustration see *Example 14*.[151]

(f). *Let us now consider the case when* α_{nn} *and* α_{1n} (correspondingly, β_{11} and β_{1n}) *form a finite, constant, real ratio.*

From (22) we see that this ratio is simply the General Circuit Parameter A, and thus the prescribed functions are: α_{11}, α_{nn} and $\frac{\alpha_{nn}}{A}$, having the real parts r_1, r_n and $\frac{r_n}{A}$. These functions are elements of a "positive real" matrix, and thus we know that:

$$\left.\begin{aligned} r_1 &\geq 0, \text{ for } \gamma \geq 0 \\ r_n &\geq 0, \text{ for } \gamma \geq 0 \\ \left(r_1 r_n - \frac{r_n^2}{A^2}\right) &= \frac{r_n}{A^2}\,(A^2 r_1 - r_n) \geq 0, \text{ for } \gamma \geq 0 \end{aligned}\right\} \quad \ldots (128)$$

[150] Naturally the special cases when Y_1 is infinite or Y_2 zero are also included.

Note: The General Circuit Parameters B and C are allowed to individually or both be zero. On the contrary, neither A nor D can in our problem be zero, which readily can be seen from (22). For A or D to be zero would, for instance, mean that in spite of the fact that no current is flowing in the entering leads of a four-terminal network, where the e.m.f. is applied, a finite current would anyhow flow through the short-circuit between the terminals at the opposite end. Or it would mean that in spite of the fact that the entering current is finite anyhow the current through the short could be infinite, which also is absurd. However, at a single frequency, corresponding to resonance frequency of a certain reactive network, A and D might become Zero.

[151] Appendix, p. 210.

The matrix, containing the General Circuit Parameters, we now split up as:

$$\left\|\begin{matrix} A & B \\ C & D \end{matrix}\right\| = \left\|\begin{matrix} A & 0 \\ 0 & \frac{1}{A} \end{matrix}\right\| \times \left\|\begin{matrix} 1 & \frac{B}{A} \\ CA & DA \end{matrix}\right\| = \left\|\begin{matrix} 1 & AB \\ \frac{C}{A} & DA \end{matrix}\right\| \times \left\|\begin{matrix} A & 0 \\ 0 & \frac{1}{A} \end{matrix}\right\| \quad \ldots(129)$$

←—Alternative I.—→ ←Alternative II.→

The matrices of the products of (129) evidently characterize a transformer and the L-structure shown in Fig. 25 respectively and vice versa for the two alternatives.

We proceed in an analogous manner as in the previous case and get:

Alternative I.

$$\left.\begin{aligned} \alpha'_{11} &= \frac{A^2D}{B} = A^2\alpha_{11} \\ \alpha'_{nn} &= \alpha'_{1n} = \frac{A}{B} = \alpha_{nn} \end{aligned}\right\} \quad \ldots\ldots(130)$$

Thus $Re\alpha'_{11}$ and $Re\alpha'_{nn}$ are each ≥ 0, for $\gamma \geq 0$, and the determinant:

$$(A^2r_1r_n - r_n^2) = r_n(A^2r_1 - r_n)$$

is also ≥ 0, for $\gamma \geq 0$, due to (128).

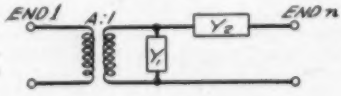


FIG. 39

Consequently we get the passive combination shown in Fig. 39, and the Y:s are simply:

$$\left.\begin{aligned} Y_1 &= AC \\ Y_2 &= \frac{A}{B} \end{aligned}\right\} \quad \ldots\ldots(131)$$

Alternative II. Here we have:

$$\left.\begin{aligned} \alpha''_{11} &= \frac{AD}{AB} = \frac{D}{B} = \alpha_{11} \\ \alpha''_{nn} &= \alpha''_{1n} = \frac{1}{AB} = \frac{\alpha_{nn}}{A^2} \end{aligned}\right\} \quad \ldots\ldots(132)$$

Thus, $Re\alpha'_{11}$ and $Re\alpha''_{nn}$ are each ≥ 0, for $\gamma \geq 0$, and the determinant:

$$\left(r_1 \frac{r_n}{A^2} - \frac{r_n^2}{A^4}\right) = \frac{r_n}{A^4}(A^2 r_1 - r_n) \text{ is also} \geq 0, \text{ for } \gamma \geq 0,$$

due to (128).

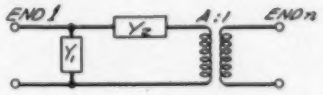


FIG. 40

Consequently we get the passive combination shown in Fig. 40, and the Y:s are simply:

$$\left.\begin{aligned} Y_1 &= \frac{C}{A} \\ Y_2 &= \frac{1}{AB} \end{aligned}\right\} \quad \ldots\ldots(133)$$

The reality clause is fulfilled and we can state:

THEOREM VIII:

If a non-singular "positive real" α- or β- matrix in such that the General Circuit Parameter A, to be obtained from this matrix, is a finite, real constant, then this specific matrix satisfies both necessary and sufficient conditions for having a network representation in either one of the structures shown in Figs. 39 or 40.

For an illustration see *Example* 15.[182]

[182] Appendix, p. 212.

(g). Now we proceed to the case *when in a "positive real" α- or β-matrix the two elements located on its principal diagonal form a finite, constant ratio.*

We now split up the matrix as:

$$\left\|\begin{matrix} A & B \\ C & D \end{matrix}\right\| = \left\|\begin{matrix} \sqrt{\frac{A}{D}} & 0 \\ 0 & \sqrt{\frac{D}{A}} \end{matrix}\right\| \times \left\|\begin{matrix} \sqrt{AD} & B\sqrt{\frac{D}{A}} \\ C\sqrt{\frac{A}{D}} & \sqrt{AD} \end{matrix}\right\| = \left\|\begin{matrix} \sqrt{AD} & B\sqrt{\frac{A}{D}} \\ C\sqrt{\frac{D}{A}} & \sqrt{AD} \end{matrix}\right\| \times \left\|\begin{matrix} \sqrt{\frac{A}{D}} & 0 \\ 0 & \sqrt{\frac{D}{A}} \end{matrix}\right\| \quad \ldots\ldots(1$$

←——— Alternative I. ———→ ←——— Alternative II. ———→

The product matrices evidently specify a transformer and a symmetrical lattice-structure, and vice versa.

The given α:s (say) are in this case: α_{11}, $\frac{A}{D}\alpha_{11}$ and α_{1n}, i.e. their real parts are r_1, $\frac{A}{D}r_1$, and r_{1n}.

We know that:

$$\left.\begin{aligned} r_1 &\geq 0, \text{ for } \gamma \geq 0 \\ \frac{A}{D}r_1 &\geq 0, \text{ for } \gamma \geq 0 \\ \left(\frac{A}{D}r_1^2 - r_{1n}^2\right) &\geq 0, \text{ for } \gamma \geq 0 \end{aligned}\right\} \quad \ldots\ldots(135)$$

Alternative I. Adding a prime to the α:s corresponding to the lattice-structure matrix, these are found to be:

$$\left.\begin{aligned} \alpha'_{11} = \alpha'_{nn} &= \alpha'_{1n}\sqrt{AD} = \frac{A}{B} = \alpha_{nn} = \frac{A}{D}\alpha_{11} \\ \alpha'_{1n} &= \frac{1}{B}\sqrt{\frac{A}{D}} = \sqrt{\frac{A}{D}}\alpha_{1n} \end{aligned}\right\} \quad \ldots\ldots(136)$$

Thus we see that $Re\alpha'_{11} = Re\alpha'_{nn}$ is ≥ 0, for $\gamma \geq 0$.
The determinant becomes:

$$\left(\frac{A^2}{D^2}r_1^2 - \frac{A}{D}r_{1n}^2\right) = \frac{A}{D}\left(\frac{A}{D}r_1^2 - r_{1n}^2\right)$$

which, due to (135), evidently also is ≥ 0, for $\gamma \geq 0$.

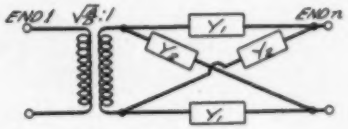


FIG. 41

Thus we get as a network representation the passive combination shown in Fig. 41. The Y:s, when expressed in the General Circuit Parameters, are simply:

$$\left.\begin{aligned} Y_1 &= \frac{1}{B}\left(A - \sqrt{\frac{A}{D}}\right) \\ Y_2 &= \frac{1}{B}\left(A + \sqrt{\frac{A}{D}}\right) \end{aligned}\right\} \quad \ldots\ldots(137)$$

Alternative II. Let the α:s corresponding to the lattice-structure matrix in this case be α'':s, and we get:

$$\left.\begin{aligned} \alpha''_{11} = \alpha''_{nn} = \alpha''_{1n}\sqrt{AD} &= \frac{D}{B} = \alpha_{11} \\ \alpha''_{1n} = \frac{1}{B}\sqrt{\frac{D}{A}} &= \sqrt{\frac{D}{A}}\alpha_{1n} \end{aligned}\right\} \quad \ldots\ldots(138)$$

Thus $Re\alpha''_{11} = \alpha''_{nn}$ is ≥ 0, for $\gamma \geq 0$.
The determinant becomes:

$$\left(r_1^2 - \frac{D}{A}r_{1n}^2\right) = \frac{D}{A}\left(\frac{A}{D}r_1^2 - r_{1n}^2\right)$$

which, due to (135), also is ≥ 0, for $\gamma \geq 0$.

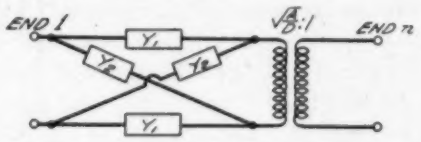


FIG. 42

Consequently we get the passive combination shown in Fig. 42, and the Y:s, expressed in the General Circuit Parameters, are:

$$\left.\begin{aligned} Y_1 &= \frac{1}{B}\left(D - \sqrt{\frac{D}{A}}\right) \\ Y_2 &= \frac{1}{B}\left(D + \sqrt{\frac{D}{A}}\right) \end{aligned}\right\} \quad \ldots\ldots (139)$$

The reality clause is fulfilled, and we can state:

THEOREM IX:

If a non-singular "positive real" α- or β-matrix is such that the ratio between the two elements which are located on its principal diagonal, corresponding to the ratio between the General Circuit Parameters A and D, is a finite, real constant,[153] *then this particular matrix satisfies both necessary and sufficient conditions for having a network representation in either one of the structures shown in Fig. 41 or 42*

For an illustration, see *Example 16*.[154]

(h). Finally we take up the case *when a "positive real" α- or β-matrix is such that a constant ratio exists between any two of the three functions.*

This is the case which we continuously meet when applying

[153] Note in this case the constant is always positive.
[154] Appendix, p. 213.

our general methods of realization to be discussed in Parts VIII, X, and XI, because when removing sets of poles from the boundary, by the aid of a partial fraction expansion, any two members of the separate component families have always a constant ratio.

The simplest structure to use for this case is *either one* of the two L-structures plus a transformer of known ratio.

In Part XI the matrix of the members of each extracted purely imaginary component family is in addition singular, and when this happens the L-structure will contain one two-terminal net work only in the form of a single reactance element.

It is evident that because functions under this item also fall under items (e), (f) and (g), any one of the structures arrived at under these items can be used as a network representation.

Further, item (g) evidently falls in between items (e) and (f).

Considering Alternative II only, say, and letting the transformer ratio D:1 as in Fig. 38 gradually become A:1 as in Fig. 40, then it can be found, that for every transformer ratio n:1, between the above mentioned limits, we get simple passive structures, while for n:s lying outside the range $1/D$ to A generally two transformers in chain connection are coming into play, and when merged together the ratio is again found to lie between the above values. Thus it is evident that the structures shown in Figs. 38 and 40 form the limit for an infinite set of equivalent networks and we can state:

THEOREM X:

If a non-singular "positive real" α- or β-matrix is such that any two of the three functions bear to each other finite, real, constant ratios, then this matrix satisfies both necessary and sufficient conditions for having a network representation in any one of an infinite number of equivalent structures, with those shown in Figs. 37 and 39 or 38 and 40 respectively as limiting cases.

For a very interesting illustration of this item, see *Example 17*.[155]

[155] Appendix, p. 215.

(i). *Standard connection diagram for the addition of the network representations of several "positive real" component α- or β-matrices.*

Besides being confronted with "positive real" α- or β-matrices which can be realized in a single step into one of the structures considered, we may meet such a "positive real" matrix which cannot be realized in this simple manner but its elements have to be split up into several components, and while each component family may separately be realized into a known structure we need to know in addition how to connect these structures together into a unit forming a total network representation of the total functions.

When considering a sum of α-families the diagram of connections would be as shown in Fig. 43, with the separate component networks connected in parallel, since they are realizations of admittance functions.

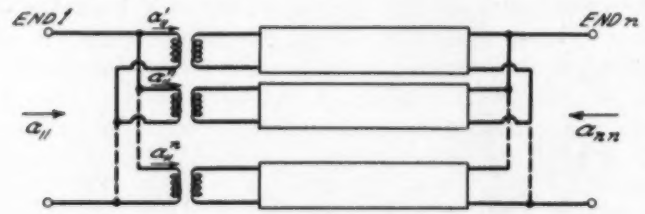


FIG. 43

When considering a sum of β-families the diagram is given by Fig. 44, with the separate component networks connected in series, since they are realizations of impedance functions.

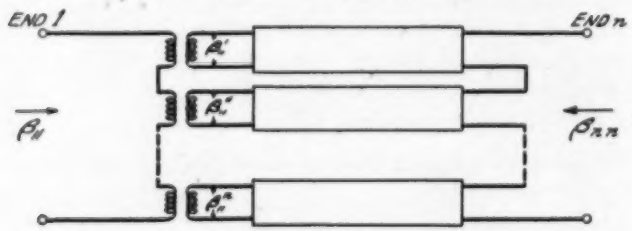


FIG. 44

From this diagram we note how the β:s add up when transferred over to the "primary" windings.

The transformers shown at one end are included *in order to force the entering and leaving currents of each component structure to be equal!* For a symmetrical lattice-component structure its transformer may usually be omitted, however.

The insertion of a 1:1 ratio transformer may also be omitted when the latter is a part of an unsymmetrical component structure if the entering and leaving currents in the component structure in question are equal without the transformer.

On the other hand, where the transformer ratio is different from 1:1 the transformer cannot be omitted.

Further it cannot be omitted if therewith some part of the network representation would be shorted out, as illustrated in some of the Examples in the following.

Where transformers are used, it is obvious that the transformer may be placed at either end of the componant structure of which it is a part.

(j). *In this final item we shall prove the fact that if two or several "positive real" α- or β-matrices are added, then the resulting matrix is also a "positive real" matrix.*

Let us, for instance, work with the α:s and assume we have the two prescribed families:

$$\left.\begin{aligned} \alpha'_{11} &= r'_1 + jx'_1 \\ \alpha'_{nn} &= r'_n + jx'_n \\ \alpha'_{1n} &= r'_{1n} + jx'_{1n} \end{aligned}\right\} \qquad \left.\begin{aligned} \alpha''_{11} &= r''_1 + jx''_1 \\ \alpha''_{nn} &= r''_n + jx''_n \\ \alpha''_{1n} &= r''_{1n} + jx''_{1n} \end{aligned}\right\} \quad \ldots\ldots(140)$$

To prove that when:

$$\left.\begin{aligned} \alpha_{11} &= \alpha'_{11} + \alpha''_{11} = (r'_1 + r''_1) + j(x'_1 + x''_1) = r_1 + jx_1 \\ \alpha_{nn} &= \alpha'_{nn} + \alpha''_{nn} = (r'_n + r''_n) + j(x'_n + x''_n) = r_n + jx_n \\ \alpha_{1n} &= \alpha'_{1n} + \alpha''_{1n} = (r'_{1n} + r''_{1n}) + j(x'_{1n} + x''_{1n}) = r_{1n} + jx_{1n} \end{aligned}\right\} . \quad (141)$$

then necessarily:

$$\left.\begin{aligned} r_1 &\geq 0, \text{ for } \gamma \geq 0 \\ r_n &\geq 0, \text{ for } \gamma = 0 \\ (r_1 r_n - r_{1n}^2) &\geq 0, \text{ for } \gamma \geq 0 \end{aligned}\right\}$$

The first two of these inequalities are fulfilled, as can be seen by inspection only.

As to the determinant condition, we get:

$$\begin{aligned} (r_1 r_n - r_{1n}^2) &= (r_1' + r_1'')(r_n' + r_n'') - (r_{1n}' + r_{1n}'')^2 = \\ &= (r_1' r_n' - r_{1n}'^2) + (r_1'' r_n'' - r_{1n}''^2) + \\ &+ (r_1' r_n'' + r_1'' r_n' - 2 r_{1n}' r_{1n}'') \end{aligned}$$

The first two terms are evidently each ≥ 0, for $\gamma \geq 0$.

This is also true regarding the last term, because adding and subtracting $2\sqrt{r_1' r_n' r_1'' r_n''}$, we have:

$$\begin{aligned} &r_1' r_n'' + r_1'' r_n' + 2\sqrt{r_1' r_n' r_1'' r_n''} - 2\sqrt{r_1' r_n' r_1'' r_n''} - \\ &- 2 r_{1n}' r_{1n}'' = (\sqrt{r_1' r_n''} - \sqrt{r_1'' r_n'})^2 + 2(\sqrt{r_1' r_n' r_1'' r_n''} - r_{1n}' r_{1n}'') \end{aligned}$$

Note here that: $(\sqrt{r_1' r_n' r_1'' r_n''} - r_{1n}' r_{1n}'')$ is ≥ 0, because $\sqrt{r_1' r_n'} \geq r_{1n}'$ and $\sqrt{r_1'' r_n''} \geq r_{1n}''$.

The reality clause and the residue requirements, in case of poles on the boundary, are met with by the α:s, because they are met with by the component α:s of (140). Further, the proof, being the same in case of β:s, can evidently be extended to addition of several "positive real" matrices and thus we can state:

THEOREM XI:

If several "positive real" α- or β-matrices are added, then the resulting matrix is also a "positive real" matrix.

The truth of this theorem is obvious if the positive definite quadratic forms associated with the inherent features of a "positive real" matrix are taken into account. From physical point of view the truth is by item (i) also self-evident, after later having proved that every "positive real" matrix is relizable into a passive structure.

Part VIII

GENERAL METHODS OF REALIZATION OF ANY "POSITIVE REAL" α- OR β-MATRIX HAVING SUCH PROPERTIES, THAT IT CAN CHARACTERIZE A PASSIVE FOUR-TERMINAL NETWORK HAVING TWO KINDS OF ELEMENTS ONLY, ETC.

(a). We now have material enough for a complete solution of the above specific cases and start with the one, when the prescribed matrix is found to have a network-representation in a purely reactive four-terminal network.

In his Reactance-Theorem (20) Foster showed that every scalar function $f_{(\lambda)}$ can be considered as being the impedance function of a passive purely reactive two-terminal network, provided all zeros and poles lie on the imaginary axis in the λ-plane and there separate each other, i.æ. alternate. Such a function or its reciprocal can always be expanded into partial fractions, and each term can be given a physical interpretation, whereupon these are joined together in a proper way.

In our case more conditions must be satisfied, but the reality clause and the residue conditions embodied in the "positive real" matrix definition[146] are found to be both necessary and sufficient.

The elements of a "positive real" α- or β-matrix falling under this item have each the general form as given by (2), rewritten below:

$$f_{(\lambda)} = \frac{a_0\lambda^{2n} + a_2\lambda^{2n-2} + a_4\lambda^{2n-4} + \ldots\ldots + a_{2n}}{b_1\lambda^{2n-1} + b_3\lambda^{2n-3} + \ldots\ldots\ldots + b_{2n-1}\lambda} \quad \ldots(2)$$

[146] See Theorem V, p. 88.

After removal of the pole at ∞, (2) goes over into:

$$f_{(\lambda)} = \frac{1}{b_1}\left[a_0\lambda + \frac{(b_1a_2 - a_0b_3)\,\lambda^{2n-2} + (b_1a_4 - a_0b_5)\,\lambda^{2n-4} + \ldots .}{b_1\lambda^{2n-1} + b_3\lambda^{2n-3} + b_5\lambda^{2n-5} + \ldots\ldots + b_{2n-1}\lambda}\right] \ldots (142)$$

Because λ in the numerator is in a lower power than in the denominator, the rational fractional part of (142) can now directly be expanded into partial fractions, and when denoting the coefficients as shown below, we get:

$$f_{(\lambda)} = k\lambda + \frac{k_0}{\lambda} + \frac{k_1 + jk_1'}{\lambda + j\omega_1} + \frac{k_1 + jk_1'}{\lambda - j\omega_1} + \frac{k_2 + jk_2'}{\lambda + j\omega_2} + \cdots \quad \ldots(143)$$

which can be simplified into:

$$f_{(\lambda)} = k\lambda + \frac{k_0}{\lambda} + \frac{2k_1\lambda + 2\omega_1k_1'}{\lambda^2 + \omega_1^2} + \frac{2k_2\lambda + 2\omega_2k_2^1}{\lambda^2 + \omega_2^2} + \cdots \quad \ldots(144)$$

Because the numerators of the fractions of (143) are found to be nothing but residues of $f_{(\lambda)}$ at respective poles (located at ∞, the origin, $\pm\omega_1$, $\pm\omega_2$, $\pm\omega_3$, . . . etc.), and because these residues, due to our necessary residue conditions, have to be real, we see that *all k′:s must be zero,*[157] and (144) goes over into:

$$f_{(\lambda)} = k\lambda + \frac{k_0}{\lambda} + \frac{2k_1\lambda}{\lambda^2 + \omega_1^2} + \frac{2k_2\lambda}{\lambda^2 + \omega_2^2} + \frac{2k_3\lambda}{\lambda^2 + \omega_3^2} + \cdots \quad \ldots(145)$$

where k, k_0, k_1, k_2, etc. are the residues of $f_{(\lambda)}$ at respective poles along the boundary.[158]

[157] Note, this offers a distinct difference from an ordinary partial fraction expansion, and demonstrates explicitly that the real part along the boundary of every term in (144) is zero. Why the numerators in (143) are residues, becomes clear when expanding any term into a Laurent's series. In this expansion we will find that the only term which contains a *negative* exponent is the one which as numerator has the expression contained as numerator of the term we are expanding. Thus this numerator is nothing but the residue.

[158] Note here the previously mentioned fact, that the coefficient (in the numerator) of a removed pole *at the origin or at* ∞ *is identical with the residue* of the function at the pole in question, while the corresponding coefficient of a *conjugate pair of removed poles* from the boundary is equal to *twice the residue* of the function at either one of the conjugate pair of poles.

As $f_{(\lambda)}$ may stand for any one of the three α:s or β:s, and as each element of the same matrix generally has the same denominator, a similar expansion as (145) is possible for each one of them and we get when considering the α:s, say:

$$\left.\begin{aligned} \alpha_{11} &= k\lambda + \frac{k_0}{\lambda} + \frac{2k_1\lambda}{\lambda^2+\omega_1^2} + \frac{2k_2\lambda}{\lambda^2+\omega_2^2} + \cdots\cdots\cdots \\ \alpha_{nn} &= k'\lambda + \frac{k_0'}{\lambda} + \frac{2k_1'\lambda}{\lambda^2+\omega_1^2} + \frac{2k_2'\lambda}{\lambda^2+\omega_2^2} + \cdots\cdots\cdots \\ \alpha_{1n} &= k''\lambda + \frac{k_0''}{\lambda} + \frac{2k_1''\lambda}{\lambda^2+\omega_1^2} + \frac{2k_2''\lambda}{\lambda^2+\omega_2^2} + \cdots\cdots\cdots \end{aligned}\right\} \quad \ldots\ldots(146)$$

Here as well as in the following when meeting similar expansions of the three α:s or β:s the different columns, i.e.

$$k\lambda,\ k'\lambda,\ k''\lambda;\ \frac{k_0}{\lambda},\ \frac{k_0'}{\lambda},\ \frac{k_0''}{\lambda};\ \frac{2k_1\lambda}{\lambda^2+\omega_1^2},\ \frac{2k_1'\lambda}{\lambda^2+\omega_1^2},\ \frac{2k_1''\lambda}{\lambda^2+\omega^2};\ \text{etc. are}$$

called component families, and the network representation of such a component family is called a component network.

We now immediately see that (146) has a network representation in a structure as shown in Fig. 43, and if (146) had been the expansions of the three β:s the network representation would have been a structure as shown in Fig. 44, although in both cases any one of the component networks may happen to be a two-terminal network.

The structure in question whether in accordance with Fig. 43 or 44, can readily be built because any two members of each component family form a finite, real, constant ratio, i.e. each corresponding matrix has a network representation in any one of an infinite number of equivalent structures, according to item (h) Part VII.

Further, these component structures are all passive because the conditions of a "positive real" matrix, derived from the consideration of such structures, are satisfied by the total

functions and then they are automatically also satisfied by the members of each component family.

This is so, because when solving for the residue of the total function at a certain pole *all other terms, except that specific one containing the pole under consideration, become zero.* Consequently by the aid of the material discussed in this item and knowing that (zeros and) poles on the boundary have their physical interpretation in reactance elements, we can state:

THEOREM XII:

If a "positive real" α- or β-matrix is such that all zeros and poles of its elements lie on the imaginary axis[159] *in the λ-plane, then this particular matrix satisfies both necessary and sufficient conditions for having a network representation in a purely reactive structure, in a combination as shown in Fig. 43 or 44 respectively.*

For an illustration, see *Example 18.*[160]

(**b**). The above method of realization can readily be extended to all those "positive real" α- or β-matrices[161] whose elements are such that all their poles lie on the negative real axis, where the zeros of the elements located on the principal diagonal also lie. These zeros and poles mutually separate each other.[162] On the other hand, the zeros of the elements located on the "secondary" diagonal lie anywhere on the real axis.

First consider the case when the sequence of zeros and poles of the elements located on the principal diagonal of a "positive real" α- or β-matrix is proceeding from and including the origin: zero – pole – zero – pole – etc.

[159] Note, the additional requirement that zeros and poles of the elements located on the principle diagonal have separation property, is contained in items (ii) and (iii) of Theorem V, specifying a "positive real" matrix.

[160] Appendix, p. 219.

[161] Note the fact that a "positive real" *α-matrix* falling under items (b) or (c) has a network representation containing a different combination of elements from the network representation of a "positive real" *β-matrix* falling under the same items.

[162] Such a function we know specifies a dissipative network containing only two kinds of elements. Bibliogr. 22 and 38.

Let us start out from function (2), rewritten under item (a), and transform it into a form corresponding to the case considered here.

Function (2) has as it stands a pole at the origin (as well as at ∞), i.e. if being a driving-point function the neighboring roots are a conjugate pair of zeros, then comes a conjugate pair of poles, etc.

When expanded into partial fractions according to (145), and when multiplied by λ on both sides of the equal sign we have:

$$\lambda f_{(\lambda)} = k\lambda^2 + k_0 + \frac{2k_1\lambda^2}{\lambda^2 + \omega_1^2} + \frac{2k_2\lambda^2}{\lambda^2 + \omega_2^2} + \cdots \qquad \ldots(147)$$

The difference between (145) and (147) is that the former is a "positive real" function, while the latter is not, due to the fact that after multiplication by λ the two zeros nearest the origin are no longer separated by a pole; moreover the pole at infinity is not simple.

By substituting $\lambda^2 = \lambda'$, however, and writing A:s in place of k:s and c:s in place of ω^2:s, we have:

$$\lambda f_{(\lambda)} = A\lambda' + A_0 + \frac{2A_1\lambda'}{\lambda' + c_1} + \frac{2A_2\lambda'}{\lambda' + c_2} + \cdots \qquad \ldots(148)$$

By these two procedures we have removed the pole from the origin and have moved all zeros and poles from the imaginary axis to the negative real axis thereby reducing their number by half, as indicated in Fig. 45.[163] If (145) stands for a driving-point function of a purely reactive structure, then it is clear from Fig. 45 that the separation property is still fulfilled after the above transformations have been done, i.e. (148) is a driving-point function of a dissipative structure having two kinds of elements only.

As the sequence of zeros and poles on the negative real axis

[163] Note the zero nearest the origin may be located in the origin itself.

is, proceeding from the origin: zero – pole – zero – pole, etc. we know that (148), as a driving-point admittance function, may specify a network having resistance and capacitance elements only; as a driving-point impedance function it may specify a network having resistance and inductance elements only.[164]

Note that the constant A_0 is nothing more than the ratio of the constant terms of the numerator and denominator of the original function before expansion. Naturally, if there is no constant term in the numerator of the original function A_0 will not

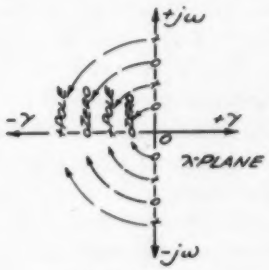


FIG. 45

appear. Further note the form of the fractions in the expansion. The function must, in this case, be expanded in the manner shown.

The transfer function, being each one of the elements on the "secondary" matrix-diagonal, can be handled in an analogous manner, in which case the separation property before as well as after the substitution may or may not be fulfilled, and further, some zeros may place themselves on the positive real axis.

$\lambda f_{(\lambda)}$ is our new function, and as all α:s or β:s can be handled in an analogous manner we get, when dropping the prime of λ and considering a set of α:s (say), the system of equations:

[164] Bibliogr. 22, p. 367. Bibliogr. 38, p. 5.

$$\left.\begin{aligned}
\alpha_{11} &= A\lambda + A_0 + \frac{2A_1\lambda}{\lambda + c_1} + \frac{2A_2\lambda}{\lambda + c_2} + \cdots\cdots\cdots \\
\alpha_{nn} &= B\lambda + B_0 + \frac{2B_1\lambda}{\lambda + c_1} + \frac{2B_2\lambda}{\lambda + c_2} + \cdots\cdots\cdots \\
\alpha_{1n} &= C\lambda + C_0 + \frac{2C_1\lambda}{\lambda + c_1} + \frac{2C_2\lambda}{\lambda + c_2} + \cdots\cdots\cdots
\end{aligned}\right\} \quad (149)$$

The A:s, B:s, C:s and c:s in (149) are numerically the same as the k:s k':s, k'':s and ω^2:s respectively of (146) although their significance is changed, because of the A:s, B:s and C:s, only those of the left component family ($A\lambda$, $B\lambda$, $C\lambda$) are residues. The others are constants only. Further, while the ω:s represented conjugate points along the imaginary axis, the c:s are single points along the negative real axis.

This, however, does not violate the fact that equations (149), like (146) always are realizable into a combination of standard structures to be connected according to the diagram 43, as only the variable is affected by the performed transformations but not the constants. In this case, however, resistance and capacitance elements only, together with transformers, are contained in the network representation to be obtained from the above expansions.

Should (149) be a set of expanded β:s, then the network representation would be in accordance with diagram 44, and resistance and inductance elements only, plus transformers, would be contained in the network.

In both cases any one of the component structures may happen to be a two-terminal network. This occurs when, in any component family, the terms corresponding to the transfer function and one of the driving-point functions are both zero.

For an illustration, see *Example 19*.[165]

(c). Let us now consider the case when the sequence of zeros and poles of the elements located on the principal diagonal of a

[165] Appendix, p. 220.

"positive real" α- or β-matrix is, proceeding from and including the origin[166]: pole – zero – pole – zero – etc., along the negative real axis.

Again we start out from function (2) and transform it into a form corresponding to the case considered here. Thus, in this case divide the expansion of (2), i.e. (145), by λ on both sides of the equality sign.

Then substitute $\lambda^2 = \lambda'$ and write A:s in place of k:s and c:s in place of ω^2:s, and we get:

$$\frac{f_{(\lambda)}}{\lambda} = A + \frac{A_0}{\lambda'} + \frac{2A_1}{\lambda' + c_1} + \frac{2A_2}{\lambda' + c_2} + \cdots \qquad \ldots(150)$$

If (2) stands for a driving-point function of a purely reactive structure, then it follows that the sequences of zeros and poles of (150) along the negative real axis is proceeding from and including the origin: pole – zero – pole – zero, etc., i.e. reversed order to that shown in Fig. 45.

Such functions, if driving-point admittances, we know may specify a network having resistance and inductance elements only. If driving-point impedances, they may specify a network having resistance and capacitance elements only.[167]

The transfer can be handled in an analogous manner with corresponding remarks as under item (b).

Thus $\frac{f_{(\lambda)}}{\lambda}$ is our new function, and dropping the prime and expanding all three α:s (say) in the same way, we obtain:

$$\left.\begin{aligned} \alpha_{11} &= A + \frac{A_0}{\lambda} + \frac{2A_1}{\lambda + c_1} + \frac{2A_2}{\lambda + c_2} + \cdots\cdots \\ \alpha_{nn} &= B + \frac{B_0}{\lambda} + \frac{2B_1}{\lambda + c_1} + \frac{2B_2}{\lambda + c_2} + \cdots\cdots \\ \alpha_{1n} &= C + \frac{C_0}{\lambda} + \frac{2C_1}{\lambda + c_1} + \frac{2C_2}{\lambda + c_2} + \cdots\cdots \end{aligned}\right\} \ldots(151)$$

[166] The pole nearest the origin may be located in the origin itself.
[167] Bibliogr. 22, p. 360. Bibliogr. 38, p. 5.

Here a similar discussion as given under item (b) applies, but note that in this case the numerators of all fractions containing λ in the denominator are residues. Also note that here the constant terms (A, B, C) are simply the ratios of the coefficients of the highest equal powers of λ in the numerator and denominator of the original functions. It is clear that if α:s, the functions characterize a four-terminal network having resistance and inductance elements only, together with transformers, and if β:s then the functions of (151) characterize a four-terminal network having resistance and capacitance elements only, plus transformers.

If α:s, the total network representation will substantially be in accordance with the diagram in Fig. 43, and if β:s, substantially in accordance with the diagram in Fig. 44, although any one of the component networks may happen to be a two-terminal network, as under the proceeding item (b).

For an illustration see *Example 20.*[168]

With the content in the last two items in mind we can now state:

THEOREM XIII:

If a "positive real" α- or β-matrix is such that the zeros and poles of the elements located on its principal diagonal lie on the negative real axis and have there separation property;[169] *the poles of the elements located on the "secondary" diagonal lie also there, while zeros of these latter elements lie anywhere along the real axis, then this particular matrix satisfies both necessary and sufficient conditions for having a network representation in a dissipative four-terminal network having two kinds of elements only.*

(**d**). It is interesting to note that equations (146) remain the same whether they stand for admittance or impedance functions

[168] Appendix, p. 222.

[169] It should be pointed out that for being realizable into a structure having two kinds of elements only the sequence of zeros and poles, must be the same for both driving point functions.

of a purely reactive network, while equations (149) if admittance functions specify a dissipative four-terminal network having resistance plus capacitance elements only, but if impedance functions they specify a four-terminal network having resistance plus inductance elements only, and vice versa regarding equations (151).

Further, due to the sequence order of zeros and poles (of the driving-point functions) along the negative real axis as mentioned under items (b) and (c),—and further due to the physical features of the different kinds of network elements, it is clear that:

(1) Our driving point α:s of a network having *resistance plus capacitance elements only*, cannot have a pole at the origin (corresponding to $\omega = 0$) but may or may not have a zero there.

The physical interpretation of this statement would be that the entering currents at zero frequency can never become infinite in a resistance-capacitance network, but may become zero in case a condenser appears in either one of the "entering" leads.

(2) Further such α:s cannot have a zero at ∞ (corresponding to $\omega = \infty$) but may or may not have a pole there.

The physical interpretation is here the fact that the entering currents (in the network under consideration) at an infinitely high frequency can never become zero, because all condensers can now be considered as shorted,—and if such a "shorted" condenser is bridged across the terminals the current for $\omega = \infty$ becomes infinite.

(3) On the other hand, when referring to *a network having resistance plus inductance elements only*, our driving-point α:s cannot have a zero at the origin but may or may not have a pole there.

The physical interpretation is again more or less self-evident, because it says that for zero frequency the entering currents in a network of above character can never be zero, but may become infinite in case an inductance coil is bridged across the terminals.

(4) Finally driving-point α:s of a network of this latter character, cannot have a pole of ∞, but may or may not have a zero there.

The corresponding physical interpretation is simply that at an infinitely high frequency the entering currents in a resistance-inductance network can never become infinitely large, but may become zero in case an inductance coil happens to appear in either one of the "entering" leads.

Part IX

REALIZATION OF CERTAIN "POSITIVE REAL" MATRICES BY A SPLITTING PROCEDURE OF THE NUMERATORS OF THE REAL PARTS OF THE MATRIX ELEMENTS; A SURVEY OF AN ALGEBRAIC GENERATION METHOD.

It remains to find a general method of finding the network representation of any "positive real" matrix, found to specify a four-terminal network having two or all three kinds of elements, but we shall first, in this Part, rather briefly outline a useful method of finding the network representation of certain types of such "positive real" matrices. When the numerator of the real part of each matrix element contains ω^2 to at most the first power,[170] then we shall show that this matrix can *always* be realized by splitting up said numerators individually, and after the corresponding λ-functions are found,[171] each component family is taken care of by one of the structures discussed in Parts VI and VII.

When the power of ω^2 in the numerator of the real parts is higher than the first, then the splitting procedure, however, does not represent a general method of finding the network representation of corresponding "positive real" matrix, but for a splitting to be advantageous additional specific features must be met with.

(**a**). Let us first consider a "positive real" α-matrix, say, having the elements:

[170] Note, odd powers of ω never occur in the *real* part.
[171] Most convenient by help of Brune's algebraic method; Bibliogr. 38, pp. 97-98.

$$\left.\begin{aligned} \alpha_{11} &= \frac{a_1\lambda + a_0}{b_1\lambda + b_0} \\ \alpha_{nn} &= \frac{a_1'\lambda + a_0'}{b_1\lambda + b_0} \\ a_{1n} &= \frac{a_1''\lambda + a_0''}{b_1\lambda + b_0} \end{aligned}\right\} \quad \ldots\ldots (152)$$

In this case we need not even solve for the real parts, but split up the α:s of (152) themselves, as:

$$\left.\begin{aligned} \alpha_{11} &= \frac{a_1\lambda}{b_1\lambda + b_0} + \frac{a_0}{b_1\lambda + b_0} = \alpha_{11}' + \alpha_{11}'' \\ \alpha_{nn} &= \frac{a_1'\lambda}{b_1\lambda + b_0} + \frac{a_0'}{b_1\lambda + b_0} = \alpha_{nn}' + \alpha_{nn}'' \\ \alpha_{1n} &= \frac{a_1''\lambda}{b_1\lambda + b_0} + \frac{a_0''}{b_1\lambda + b_0} = \alpha_{1n}' + \alpha_{1n}'' \end{aligned}\right\} \quad \ldots\ldots (153)$$

From Part II we know that zeros and poles of the determinant of the α:s of (152) all lie in the left half of the λ-plane. This fact on the other hand forces the two determinants $(a_1a_1' - a_1''^2)$ and $(a_0a_0' - a_0''^2)$ to be $\geq$ zero individually, because in the α-determinant the former is nothing but the coefficient of λ^2 in the numerator, and the latter is the constant term, and both are necessary $\geq$ zero.

Thus because in addition $a_1a_0a_1'$ and a_0' are each $\geq$ zero by item (e) Part II, and as the reality clause is fulfilled, we see that each component family of (153) also forms a "positive real" matrix, and they can evidently separately be realized by the aid of item (h), say, Part VII.

From our discussion above we could be inclined to think that a set of functions:

$$\left.\begin{aligned} \alpha_{11} &= \frac{a_1\lambda + a_0}{b_2\lambda^2 + b_1\lambda + b_0} \\ \alpha_{nn} &+ \frac{a_1'\lambda + a_0'}{b_2\lambda^2 + b_1\lambda + b_0} \\ \alpha_{1n} &= \frac{a_1''\lambda + a_0''}{b_2\lambda^2 + b_1\lambda + b_0} \end{aligned}\right\} \quad \ldots\ldots (154)$$

also could be split up in the same manner as shown in (153) and realized, because the determinants $(a_1a_1' - a_1''^2)$ and $(a_0a_0^1 - a_0''^2)$ are certainly individually $\geq$ zero, and again a_1a_1' a_0 and a_0' are each $\geq$ zero.

It is immediately evident by inspection only, however, that above two determinants in the case of (154) do not force the two component families to form "positive real" matrices, even if the total α:s are elements of such a matrix.

This is also clear from the fact that one component family would be:

$$\left.\begin{aligned} \alpha_{11}' &= \frac{a_0}{b_2\lambda^2 + b_1\lambda + b_0} \\ \alpha_{nn}' &= \frac{a_0'}{b_2\lambda^2 + b_1\lambda + b_0} \\ \alpha_{1n}' &= \frac{a_0''}{b_2\lambda^2 + b_1\lambda + b_0} \end{aligned}\right\} \quad \ldots\ldots (155)$$

but a necessary condition for α_{11}' and α_{nn}' to be "positive real" functions, which they have to be, is that the power of λ in the numerator and denominator does not differ by more than unity.[172]

The matrix of the functions (154) can be realized, however, if instead of splitting up the α:s, we split up their real parts.

Then each component family of real parts can be made to generate corresponding λ-functions, and as we shall have to use

[172] Bibliogr. 38, p. 30.

a similar procedure in Part XI we shall in the next item discuss the fundamental principles of the algebraic generation method, for the case when the *real* part is known.

(**b**). Suppose the function:

$$f_{(\lambda)} = \frac{g_{(\lambda)}}{h_{(\lambda)}} = \frac{a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots\ldots\ldots + a_0}{b_n\lambda^n + b_{n-1}\lambda^{n-1} + \ldots\ldots\ldots + b_0} \quad \ldots(156)$$

is regular in the right half of the λ-plane and has no poles on the boundary (if originally present they are simply removed).

The a:s and b:s are real constants. Let the real part of $f_{(\lambda)}$ along the boundary be:

$$Ref_{(j\omega)} = \frac{G_{(\omega^2)}}{H_{(\omega^2)}} = \frac{A_n\omega^{2n} + A_{n-2}\omega^{2n-2} + \ldots\ldots + A_0}{B_n\omega^{2n} + B_{n-2}\omega^{2n-2} + \ldots\ldots + B_0} \quad .(157)$$

where the A:s and B:s are real constants.

Then it is found that real roots of $H_{(\omega^2)}$ are equal to the negative squares of real roots of $h_{(\lambda)}$.

If we know the real roots of $H_{(\omega^2)}$ then, letting such a root be σ_r, the corresponding root λ_r say, of $h_{(\lambda)}$ is found from:

$$\lambda_r = -\sqrt{-\sigma_r} \qquad \ldots\ldots(158)$$

The minus sign in front of the radical simply implies that the λ_r falling on the negative real axis should be considered, and the minus sign inside the radical makes the radical itself real, *because σ_r is always negative.*

Besides negative real roots $H_{(\omega^2)}$ may, however, also contain conjugate pairs of complex roots.

In this case it is readily seen that if a pair of such roots be represented by a factor:

$$M_2\omega^4 + M_1\omega^2 + M_0 \qquad \ldots\ldots(159)$$

then the corresponding factor in $h_{(\lambda)}$ will be given by:

$$+\sqrt{M_2}\lambda^2+\left[+\sqrt{M_1+2\sqrt{M_0M_2}}\right]\lambda+\sqrt{M_0} \quad .(160)$$

In this manner we can by the aid of $H_{(\omega^2)}$, equation (158) and the expression (160) determine $h_{(\lambda)}$ but for an arbitrary, positive, constant factor.

This factor, however, is fixed by a comparison of the coefficients for the highest (say) power of ω^2 in $H_{(\omega^2)}$ and of the highest power of λ in the obtained λ-function, from:

$$B_n=\sum_{s=-\infty}^{+\infty} b_{r+s}b_{r-s}(-1)^s \qquad \ldots\ldots(161)$$

where B_n is the coefficient of the highest power of ω^2 in $H_{(\omega^2)}$, and b_r the coefficient of the highest power of λ in $h_{(\lambda)}$ to be found. The factor then to multiply the generated λ-function with, in order to obtain $h_{(\lambda)}$, is b_r divided with the coefficient for the highest power of λ in the generated λ-function.

The relation which should hold is:

$$B_n=b_n^2 \qquad \ldots\ldots (161a)$$

Now $h_{(\lambda)}$ is known and it remains to find $g_{(\lambda)}$. This, however, is simply accomplished by the aid of the equations:

$$A_r=\sum_{s=-\infty}^{+\infty} a_{r+s}b_{r-s}(-1)^s \qquad \ldots\ldots(162)^{178}$$

where each A_r is a coefficient of $G_{(\omega^2)}$ in (157), b_r is the coefficient of λ^r in $h_{(\lambda)}$ found above; a_r is the coefficient of λ^r in $g_{(\lambda)}$ we are looking for.

Now $g_{(\lambda)}$ is also found, *and $f_{(\lambda)}$ in (156) is uniquely determined.*

[178] Note: the a:s and b:s of (161) and (162) are the coefficients of (156) we are looking for. In this Part we only split up the numerators of the real parts, i.e., $h_{(\lambda)}$ is known beforehand and only $g_{(\lambda)}$ has to be found. In Part XI, however, $h_{(\lambda)}$ has also to be determined; this is why we have included all stages of the method.

The method of generation will be more clear from the illustrating *Example* 21.[174]

From this Example it is clear, that because the generated $f_{(\lambda)}$ is a "positive real" function, *the power of* ω^2 *in the numerator and denominator of the real part of such a function can differ by any number,*[175] *while the power of* λ *in the numerator and denominator of the "positive real" function itself cannot differ by more than one!*

(c). It is now clear that the α:s of (154), if elements of a "positive real" matrix, always can be realized by a procedure of splitting the numerator of their real part of each element and then letting each component family of real parts generate a corresponding component family of λ-functions.

Further, it is evident that a "positive real" α-matrix having the elements:

$$\left.\begin{aligned} \alpha_{11} &= \frac{a_2\lambda^2 + a_1\lambda + a_0}{b_1\lambda + b_0} \\ \alpha_{nn} &= \frac{a_2'\lambda^2 + a_1'\lambda + a_0'}{b_1\lambda + b_0} \\ \alpha_{1n} &= \frac{a_2''\lambda^2 + a_1''\lambda + a_0''}{b_1\lambda + b_0} \end{aligned}\right\} \quad \ldots\ldots (163)$$

can also always be realized by a splitting procedure, because the numerators of the real parts have the same form as the numerators of the real parts of (154), i.e. they contain no terms of higher power in ω than ω^2.

Moreover, it is clear that if the real parts, along the boundary, of the elements of a "positive real" α-matrix (say) are:

[174] Appendix, p. 224.

[175] Note, the power of ω^2 in the numerator of the real part of a "positive real" function must always be $\leq$ the power of ω^2 in the denominator. Another possibility for $F_{(\omega^2)}$ to become infinite (which is not allowed) for positive real ω^2 values is occurrence of poles for such values. This cannot happen, however, when the potential functions correspond to λ-functions being elements of a "positive real" matrix.

$$\left.\begin{aligned} r_1 &= \frac{A_1\omega^2 + A_0}{B_n\omega^{2n} + B_{n-2}\omega^{2n-2} + B_{n-4}\omega^{2n-4} + \ldots + B_0} \\ r_n &= \frac{A_1'\omega^2 + A_0'}{B_n\omega^{2n} + B_{n-2}\omega^{2n-2} + \ldots\ldots\ldots\ldots + B_0} \\ r_{1n} &= \frac{A_1''\omega^2 + A_0''}{B_n\omega^{2n} + B_{n-2}\omega^{2n-2} + \ldots\ldots\ldots\ldots + B_0} \end{aligned}\right\} \ldots (164)$$

where n may be any positive real number, they can always be realized, by a splitting procedure, into two standard structures, usually including transformers.

(**d**). It remains then to realize a "positive real" α-matrix (say) having such elements, that the real parts, along the boundary, have the general form:

$$\left.\begin{aligned} r_1 &= \frac{A_n\omega^{2n} + A_{n-2}\omega^{2n-2} + A_{n-4}\omega^{2n-4} + \ldots\ldots\ldots + A_0}{B_n\omega^{2n} + B_{n-2}\omega^{2n-2} + B_{n-4}\omega^{2n-4} + \ldots\ldots\ldots + B_0} \\ r_n &= \frac{A_n'\omega^{2n} + A_{n-2}'\omega^{2n-2} + A_{n-4}'\omega^{2n-4} + \ldots\ldots\ldots + A_0'}{B_n\omega^{2n} + B_{n-2}\omega^{2n-2} + B_{n-4}\omega^{2n-4} + \ldots\ldots\ldots + B_0} \\ r_{1n} &= \frac{A_n''\omega^{2n} + A_{n-2}''\omega^{2n-2} + A_{n-4}''\omega^{2n-4} + \ldots\ldots\ldots + A_0''}{B_n\omega^{2n} + B_{n-2}\omega^{2n-2} + B_{n-4}\omega^{2n-4} + \ldots\ldots\ldots + B_0} \end{aligned}\right\} . (165)$$

If the coefficients in the numerator of r_1 and r_n are all positive,[176] and if in addition the determinants of the coefficients for the same power of ω^2 in all three numerators are $\geq$ zero, i.e. $(A_nA_n' - A_n''^2) \geq 0$; $(A_{n-2}A_{n-2}' - A_{n-2}''^2) \geq 0$ . . . $(A_0A' - A_0''^2) \geq 0$, then we know from the foregoing item, that the α:s corresponding to the r:s of (165) can be realized into a number of standard structures, usually containing transformers, to be connected as shown in Fig. 43.

Another possibility of realizing (165) is to split up the numera-

[176] Or any one may be zero, but then the corresponding coefficients in the numerator of r_{1n} must also be zero.

tors in such a manner, that several terms are included in the same group, and so selected that a constant ratio exists between two members of the same component family, at the same time as items (iv) and (v) of Theorem V are satisfied by the components.

As an illustration of this latter method see *Example 22*,[177] which also brings out the way of realizing removed poles from the boundary separately. Further, the Example shows how the different component structures are joined together, according to the scheme of Fig. 43, to a total network representation of the total functions.

As a whole, it can be said that the splitting method of realization, as outlined in this Part, has theoretical importance, although it does not seem to be possible to make it general for application on any "positive real" α- or β-matrix. Further, a disadvantage is the fact that usually superfluous elements are introduced.

[177] Appendix, p. 225.

PART X

A BRIEF OUTLINE OF A GENERAL METHOD FOR FINDING A NETWORK REPRESENTATION OF ANY "POSITIVE REAL" α- OR β-MATRIX

For the case when the lowest point on the r-determinant $-\omega^2$ curve or on the h-determinant $-\omega^2$ curve respectively, falls at a finite frequency, "negative" reactance elements are introduced into the "network representation" of the prescribed matrix, however, and no method to produce the effect of these fictitious elements, by the aid of positive ones, has been found.

In the next Part, however, another far superior general method, not having this disadvantage, will be introduced, and the reasons why we include this Part are:

(1) The derivation procedures are of great interest and value.

(2) The general method considered here offers a one to one comparison with Brune's synthesis of a two-terminal network.

(3) When zeros and poles of the matrix elements lie as under items (b) and (c) of Part VIII no negative network elements come into play, and thus we have here a second general method for synthesizing of a dissipative four-terminal network containing two kinds of network elements only.

(**a**). After removal of - and separately realizing originally existing poles on the boundary, our functions are regular there also, and considering the α:s say, we get by the aid of (47) the following expression for the α-determinant along the boundary:

$$^{\circ}(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2)_{j\omega} = (r_1 r_n - r_{1n}^2) - (x_1 x_n - x_{1n}^2) + \\ + j\,(r_1 x_n + r_n x_1 - 2r_{1n}x_{1n}) \qquad \ldots\ldots(166)$$

In case equations (1)[178] had contained the β:s instead of the α:s the same discussion as to regularity and removal of inherent poles on the boundary would apply. The expansion of the β-determinant along the boundary is, by the aid of (62) :

$$(\beta_{11}\beta_{nn} - \beta_{1n}^2)_{(j\omega)} = (h_1 h_n - h_{1n}^2) - (v_1 v_n - v_{1n}^2) + \\ + j\,(h_1 v_n + h_n v_1 - 2h_{1n}v_{1n}) \qquad \ldots\ldots(167)$$

From (15) and (16) it is clear that a zero of the α-determinant corresponds to a pole of the β:s and vice versa, that is if the α- or β-determinants respectively of (166) and (167) are made equal to zero at a certain frequency by "extraction" of certain quantities from the driving-point functions, then poles on the boundary at the same ω will appear in the members of the next family after an "inversion"[179] has been performed, and as shown such poles can then be removed and separately realized into purely reactive structures.

It should be understood, however, that in case the zero of one of the two determinants is produced at the origin or at ∞, then the members of the next family, obtained by an inversion, will have a single pole at the corresponding point, while if the zero is produced at a finite frequency, then naturally the members of the next family are made to contain a conjugate pair of poles at the ω:s in question.

Considering the α-determinant of (166) say, we know that in order to become zero at a certain ω the real part and the imaginary part of said determinant must become zero simultaneously.

The real part contains one r-determinant and one x-determinant; let us make each of them equal to zero individually at a certain ω.

Extracting a certain positive constant value r_1' from r_1, making the r-determinant equal to zero at some ω, we get:

[178] P. 14.

[179] By "*inversion*" we shall in the following always mean a transformation from α-functions to β-functions and vice versa, according to the general relations (15) and (16), p. 31. Also the different tenses of the corresponding verb will be used, throughout.

$$F_{(\omega^2)} = \frac{f_{(\omega^2)}}{h^2_{(\omega^2)}} = (r_1 - r_1')r_n - r_{1n}^2 = (r_1 r_n - r_{1n}^2) - r_1' r_n \geq 0. \quad (168)$$

Instead of extracting a positive constant value r_1' from r_1 we could as well have extracted a positive constant value r_n' from r_n, but in either case the extracted constant value must be $\leq r_{1\,min}$ and $\leq r_{n\,min}$ respectively, where $r_{1\,min}$ is the value of r_1 which is obtained when substituting in it the ω^2-value at which the $r_1 - \omega^2$ curve between 0 and $+\infty$ has its lowest point, and $r_{n\,min}$ is the analogous r_n-value.[180]

After having extracted the proper r_1' value say, then $F_{(\omega^2)}$ has a zero at $\omega^2 = \omega_0^2$ say, but is ≥ 0 for $0 < \omega^2 < +\infty$, and when the found ω_0^2 value is finite, then naturally $F_{(\omega^2)}$ has a double-zero at the point in question.

That a proper r_1' value always can be found is clear from the following considerations:

Let the $r_1 - \omega^2$, $r_n - \omega^2$ and $(r_1 r_n - r_{1n}^2) - \omega^2$ curves be those shown in Fig. 46.

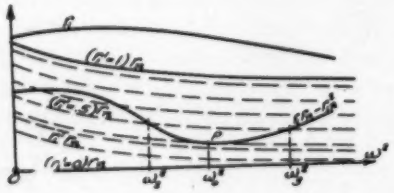


FIG. 45

Thus an infinite number of $r_1' r_n - \omega^2$ curves (or $r_n' r_1 - \omega^2$ curves in case the extraction is made from r_n) would fall as indicated in the figure, and because all values on the $r_n - \omega^2$, $(r_1 - \omega^2)$ and the $(r_1 r_n - r_{1n}^2) - \omega^2$ curves are ≥ 0 for $0 < \omega^2 < +\infty$, there is always a constant positive r_1'-value such that when multiplied with r_n, or a constant positive r_n' value such that when multiplied with r_1, this respective curve *is made to touch* the $(r_1 r_n - r_{1n}^2) - \omega^2$

[180] See the first figure in *Example* 23.

curve, at which point (P) we have our ω_0^2, making $F_{(\omega^2)}$ in (168) equal to zero at that praticular frequency.

The method evidently also gives us a tool to attack another problem, i.e. the design of a four-terminal network which fulfils the requirements inherent in the prescribed functions *within a certain frequency range,*[181] while outside this range its behavior is not controlled by those functions.

In practice both ω_0^2 and r_1' (say) can be found in a very simple manner:

From (168) we see that $(r_1r_n - r_{1n}^2) - r_1'r_n$ is ≥ 0
Thus:

$$(r_1r_n - r_{1n}^2) \geq r_1'r_n$$

We know that r_n is a quantity ≥ 0 and thus the above inequality can consequently be written:

$$\frac{r_1r_n - r_{1n}^2}{r_n} \geq r_1' \qquad \ldots\ldots(169)$$

i.e. the r_1' corresponding to the touching point between the $r_1'r_n - \omega^2$ and the $(r_1r_n - r_{1n}^2) - \omega^2$ curves in Fig. 46 *is simply the lowest value on the* $\frac{r_1r_n - r_{1n}^2}{r_n} - \omega^2$ *curve,* shown, for instance, in Fig. 47, and ω_0^2 is, for the different cases, obtained as shown in the following.

Should the lowest point of the above curve fall at the origin, then ω_0^2 is $= 0$ and r_1' is found from:

$$r_1' = \left[\frac{r_1r_n - r_{1n}^2}{r_n}\right]_{\omega^2=0} \qquad \ldots\ldots(170)$$

Should the lowest point fall at ∞, then ω_0^2 is $= \infty$ and r_1' is found from:

$$r_1' = \left[\frac{r_1r_n - r_{1n}^2}{r_n}\right]_{\omega^2=\infty} \qquad \ldots\ldots(171)$$

[181] Or frequency ranges, as indicated in the figure by the $r_1'r_n - \omega^2$ curve for $r_1' = .5$, in which case the behavior of the network is controlled for positive ω^2 — values $\leq \omega_2^2$ and $\geq \omega_3^2$.

Finally, should the lowest point fall at a finite ω^2-value; then ω_0^2 is most conveniently found from the derivative:

$$\frac{d}{d\omega^2}\left[\frac{r_1 r_n - r_{1n}^2}{r_n}\right] = 0 \qquad \ldots\ldots(172)$$

whereupon r_1' can be obtained from the equation:

$$r_1' = \left[\frac{r_1 r_n - r_{1n}^2}{r_n}\right]_{\omega^2=\omega_0^2} \qquad \ldots\ldots(173)$$

Should the curve in Fig. 47, or an analogous curve, have more than one "minimum" point, then the lowest one of such points

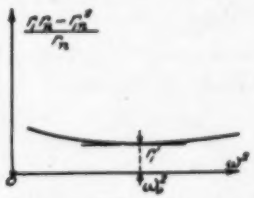


FIG. 47

should be considered, and should the curve be a straight line parallel to the ω^2-axis, then r_1' is given by the distance between said lines, and ω_0^2 should be selected equal to zero or ∞, because at these points the imaginary part of each α is zero. Should the curve fall along the ω^2-axis, then we have the case considered in the final Part of this paper.

(**b**). Then, in case ω_0^2 is finite, we have to find two constant values x_1' and x_n' such that when extracted from x_1 and x_n all three parts[182] to the right of the equal sign in (166) become equal to zero at $\omega = \omega_0$.

[182] x_1' and x_n' are the constant values extracted from x_1 and x_n in order to get a zero of the α-determinant on the boundary at $\omega^2 = \omega_0^2$. Note, it is not enough to extract a constant value (x') from one of the x:s, because this extraction must make the x-determinant as well as the imaginary part of (166) zero simultaneously. Further note an extraction from x_1 *or* x_n plus x_{1n} would have no physical interpretation. Thus, the extraction has to be made from x_1 and x_n.

Thus, we get the set of equations:

$$\left.\begin{aligned} (r_1 - r_1')\, r_n - r_{1n}^2 &= 0 \\ (x_1 - x_1')\,(x_n - x_n') - x_{1n}^2 &= 0 \\ (r_1 - r_1')\,(x_n - x_n') + r_n(x_1 - x_1') - 2r_{1n}x_{1n} &= 0 \end{aligned}\right\} \ldots (174)$$

Then, by substituting the first two equations of (174) into the last one in two different ways we get:

$$\left.\begin{aligned} [r_{1n}x_{1n} - r_n(x_1 - x_1')]_{\omega=\omega}^2 &= 0 \\ [r_{1n}(x_n - x_n') - r_n x_{1n}]_{\omega=\omega}^2 &= 0 \end{aligned}\right\}$$

And finally when solving for x_1' and x_n' we have:

$$\left.\begin{aligned} x_1' &= \left[\frac{r_n x_1 - r_{1n}x_{1n}}{r_n}\right]_{\omega=\omega} \\ x_n' &= \left[\frac{r_{1n}x_n - r_n x_{1n}}{r_{1n}}\right]_{\omega=\omega} \end{aligned}\right\} \quad \ldots\ldots (175)$$

(c). Thus we have produced a zero of the α-determinant at a finite frequency on the boundary and are ready to start a series of inversions according to (15) and (16), from which equations it is clear that therewith the power of λ should double for every time we invert.

As demonstrated with the α'- and β''-determinant equations of the next item, however, and which also readily can be proved,[183] the numerator of the determinant of the prescribed matrix, when made to have a zero or a conjugate pair of zeros on the boundary,[184] after said zero(s) has (have) been factored out, becomes a factor in the numerator of the determinant of that inverse component matrix whose elements are what remains[185] after produced poles on the boundary have been removed.[186]

[183] The proof can be found in the author's doctors dissertation in the Vail-library at the Massachusetts Institute of Technology.

[184] See (176).

[185] See (177).

[186] When the determinant of the real parts is equal to zero, then a cancellation takes place already in the first inversion, as will be shown in the next Part.

(**d**). We now are ready to show how our functions can be exhausted of all their zeros and poles.

Thus, starting out from a "positive real" α-matrix say, whose elements, after removal of originally existing poles on the boundary, are of the $\frac{n^{th}}{n^{th}}$ degree in λ (i.e. the numerators as well as the denominators are of the n^{th} degree) and for which ω_0 comes out finite, we have, by the aid of (1):

$$\left.\begin{aligned} \alpha_{11} &= r_1' + \frac{x_1'}{\omega_0}\lambda + \frac{g_{1(\lambda)}'}{h_{(\lambda)}} \\ \alpha_{nn} &= \frac{x_n'}{\omega_0}\lambda + \frac{g_{n(\lambda)}'}{h_{(\lambda)}} \\ \alpha_{1n} &= \frac{g_{1n(\lambda)}}{h_{(\lambda)}} \end{aligned}\right\} \quad \ldots\ldots (176)^{187}$$

where $g_{1(\lambda)}'$ and $g_{n(\lambda)}'$ are each of one degree higher in λ than $g_{1(\lambda)}$ and $g_{n(\lambda)}$ in (1), because the introduced poles at ∞ in α_{11} and α_{nn} have to be "balanced" for by similar poles having the same residues but of opposite sign.

Letting the component family to the right be α':s, then we see that the determinant $(\alpha_{11}'\alpha_{nn}' - \alpha_{1n}'^2)$ is of the $\frac{(2n+2)^{th}}{2n^{th}}$ degree.

Further, because ω_0 here is finite, this determinant must contain $(\lambda^2 + \omega_0^2)$ as a factor in the numerator, and the remaining part $\Delta_{(\lambda)}$ should become a factor in the numerator of the determinant of the next matrix after a conjugate pair of poles, contained in each element, have been removed from the boundary.

Thus we can write:

$$(\alpha_{11}'\alpha_{nn}' - \alpha_{1n}'^2) = \frac{(\lambda^2 + \omega_0^2)\,\Delta_{(\lambda)}}{h_{(\lambda)}^2}$$

[187] Let us for the present assume that the introduced poles in (176) have negative residues, thus leaving $\frac{g_1'^{(\lambda)}}{h_{(\lambda)}}$ and $\frac{g_n'^{(\lambda)}}{h_{(\lambda)}}$ as "positive real" functions, although this is not always the case.

and forming the corresponding β':s we get:

$$\left.\begin{aligned}
\beta'_{11} &= \frac{g'_{n(\lambda)}h_{(\lambda)}}{(\lambda^2+\omega_0^2)\,\Delta_{(\lambda)}} = \frac{2k_1\lambda}{\lambda^2+\omega_0^2} + \frac{g''_{1\,(\lambda)}}{\Delta_{(\lambda)}} \\
\beta'_{nn} &= \frac{g'_{1(\lambda)}h_{(\lambda)}}{(\lambda^2+\omega_0^2)\,\Delta_{(\lambda)}} = \frac{2k_n\lambda}{\lambda^2+\omega_0^2} + \frac{g''_{n\,(\lambda)}}{\Delta_{(\lambda)}} \\
\beta'_{1n} &= \frac{g_{1n(\lambda)}h_{(\lambda)}}{(\lambda^2+\omega_0^2)\,\Delta_{(\lambda)}} = \frac{2k_{1n}\lambda}{\lambda^2+\omega_0^2} + \frac{g''_{1n(\lambda)}}{\Delta_{(\lambda)}}
\end{aligned}\right\} \quad \ldots\ldots (177)$$

where the k:s are the residues of the β':s at either one of their conjugate pair of poles.

Here it is clear that the β':s are of the $\frac{(2n+1)^{th}}{(2n+2)^{th}}$ degree, and consequently the members of the component family to the right (β'':s say) are of the $\frac{(2n-1)^{th}}{(2n)^{th}}$ degree, i.e. they have a zero at ∞ making it possible to remove a pole at that "point" from the members of the next family.

Forming the β''-determinant:

$$(\beta''_{11}\beta''_{nn} - \beta''^2_{1n}) = \frac{\Delta_{(\lambda)}\Delta_{1(\lambda)}}{\Delta^2_{(\lambda)}} = \frac{\Delta_{1(\lambda)}}{\Delta_{(\lambda)}}$$

and then forming the corresponding α'':s we get:

$$\left.\begin{aligned}
\alpha''_{11} &= \frac{g''_{n\,(\lambda)}}{\Delta_{1(\lambda)}} = k'_1\lambda + \frac{g'''_{1\,(\lambda)}}{\Delta_{1(\lambda)}} \\
\alpha''_{nn} &= \frac{g''_{1\,(\lambda)}}{\Delta_{1(\lambda)}} = k'_n\lambda + \frac{g'''_{n\,(\lambda)}}{\Delta_{1(\lambda)}} \\
\alpha''_{1n} &= \frac{g''_{1n(\lambda)}}{\Delta_{1(\lambda)}} = k'_{1n}\lambda + \frac{g'''_{1n\,(\lambda)}}{\Delta_{1(\lambda)}}
\end{aligned}\right\} \quad \ldots\ldots (178)$$

where the k':s are the residues of the α'':s at their pole at ∞.

In (178) the α'':s are of the $\frac{(2n-1)^{th}}{(2n-2)^{th}}$ degree, and thus the members of the component family to the right (α''':s say) are of the $\frac{(2n-2)^{th}}{(2n-2)^{th}}$ degree.

The α'''-determinant is, after cancellation of $\Delta_{1(\lambda)}$, also of the $\frac{(2n-2)^{th}}{(2n-2)^{th}}$ degree, and *thus the inversions have now been carried through a complete cycle.*

The determinant of the α:s, from which we started out, was of the $\frac{2n'^{th}}{2n'^{th}}$ degree, and consequently we see that *in one cycle the power of* λ, *in the numerator as well as in the denominator, is*

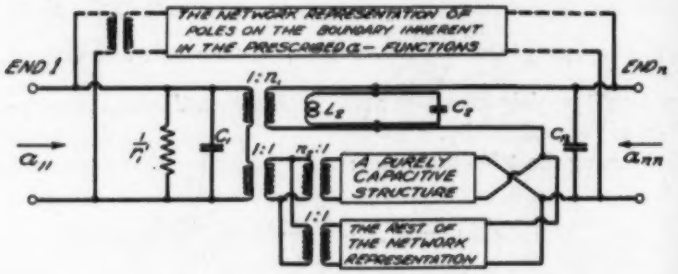


FIG. 48

decreased by two units, and the same decrease takes place in every cycle, until the given functions generally are exhausted of all their zeros and poles.

The "network representation," for each cycle, when starting out from a set of α:s, would then be as shown in Fig. 48.

On the other hand if starting out from a set of β:s the "network representation" would be as shown in Fig. 49.

For an illustration of the above outlined general method see *Example* 23.[188]

[188] Appendix, p. 234.

It may be added that instead of starting out with introducing two poles at ∞, as shown in equation (176), we could as well have introduced two poles at the origin, getting:

$$\left.\begin{aligned} \alpha_{11} &= r_1' + \frac{\omega_0 x_1'}{\lambda} + \frac{g_{1(\lambda)}'}{h'_{(\lambda)}} \\ \alpha_{nn} &= \frac{\omega_0 x_n'}{\lambda} + \frac{g_{n(\lambda)}'}{h'_{(\lambda)}} \\ \alpha_{1n} &= \frac{g_{1n(\lambda)}}{h_{(\lambda)}} \end{aligned}\right\} \quad \ldots\ldots (179)$$

where the two upper members of the component family to the right are made to "balance" for the introduced poles again, but

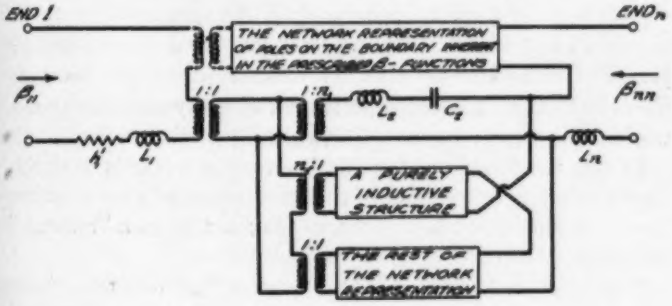


FIG. 49

they naturally are different from those of the corresponding family in (176).

Then we proceed in an analogous manner with inversions etc., and again we will find that in each cycle the power in λ goes down to the same extent as already shown, although naturally the "network representation" will be different.

During the evaluation procedure the conponent family to the right may, however, happen to form a singular matrix. In such a case the functions cannot be exhausted of all their zeros

and poles as outlined above, but, as shown in the end of the next Part, every singular "positive real" matrix is realizable into a structure comprising one two-terminal network only, and this structure, except when all the matrix elements are equal, contains in addition a transformer.

(e). In the evaluation procedure in the previous item invariably in each cycle, for which ω_0 comes out finite, poles having negative residues are coming into play, and, as such poles of a driving-point function should correspond to negative reactance elements, the "feeling" at the terminals due to these elements has to be produced by means of positive ones, if the total structure is to be passive, connected for instance in such a manner that their mutual inductance is negative.

Even if the effect of the introduced fictitious negative reactance elements could be produced, not less than *four different cases are possible*, obtained for different sign combinations of the two residues of the first introduced poles, and as many separate proofs would be necessary to show the generality of the method.

In case the residue of the first[189] introduced pole is positive, then a pole having a residue of the same magnitude but negative has to be introduced in the rest of the function as a "balance" necessity.

Thus this "rest function" is no longer a "positive real" function. Later[190] a pole at the same point on the boundary and having negative residue will "fall out," however, and when ready to start the next cycle our functions should again be elements of a "positive' real" matrix.

To find a method of connections in order to produce the effect (as "felt" at the terminals) of the fictitious negative reactance elements, by the aid of "available" positive ones, may be indicated by starting out from the lower four-terminal network in

[189] See (176) or (179). Holds for either one or both of the driving-point functions.
[190] See (178).

Fig. 48 or 49 and go backwards solving for the functions characterizing the total structure at its main terminals.

Then *by equating such terms to zero, which are not allowed to appear in those functions*, we obtain certain relationships between "positive" and "negative" reactance elements. Then if we can prove that these found relationships always can be met with in practice we are safe.

Thus considering Fig. 49 say, we first add the impedances of the two lower component structures. Then we transform them into corresponding admittances by the aid of (15). Then these admittances are added to the admittances of the upper component structure containing the transformer and the resonance-component. Then we transform our so obtained admittances into impedances by the aid of (16). Then we add the two poles corresponding to L_1 and L_n, and after adding[191] h_1' we ought to get the β:s at the terminals.

(f). It is interesting to make a comparison between Brune's general method of finding a network representation of a single "positive real" function, and our general method (so far complete as outlined in this Part) of finding a network representation of a "positive real" matrix, as analogies exist throughout:

(1) Brune's single function is regular in the right half of the λ-plane, making it possible for him to limit his efforts to the boundary.

Each one of our functions is regular in the same region, and we also can deal with the boundary only.

(2) First Brune removed all originally existing zeros and poles on the boundary and realized them into single reactance elements.

We cannot always remove all zeros on the boundary from each element of our prescribed matrix, but we remove all poles there and realize them into two-terminal or four-terminal reactance networks.

[191] The adding of h_1' will not affect those terms which are to be equated to zero, however, nor will it introduce any new powers of λ.

(3) Then Brune produced new zeros on the boundary by extracting a constant "minimum" value from the real part of his function and introducing a pole at ∞ (having positive or negative residue depending upon the shape of the imaginary part versus ω-curve) making it generally possible to remove a conjugate pair[192] of poles from the reciprocal of his impedance function.

When the determinant of our prescribed matrix has no more zeros or poles on the boundary we produce new zeros there by extracting a certain constant minimum value from a ratio of the determinant of the real parts and the real part of either one of the two driving-point functions, and introducing two poles at ∞ (say) having positive or negative residues,—or perhaps one has positive and the other one has negative residue. Then a conjugate pair of poles can generally be removed from each member of the next family.

(4) After again having inverted the rest of the reciprocal of his function, i.e. when again having an impedance function, Brune could remove a pole at ∞ having a residue of opposite sign to the one first introduced.

After having inverted for the second time, i.e. being back to a matrix containing the same kind of functions as those we started out from, we can generally remove a pole at ∞ from each function, and the residues of those removed from the two driving-point functions have opposite signs to those first introduced in the corresponding functions.

(5) In all steps the quantities Brune introduced and removed respectively were represented by single elements (in the second step usually by a resonance component).

We generally have to give our introduced and removed quantities, respectively, physical interpretation in two-terminal and four-terminal component networks.

[192] This is true only when $\mathrm{Re}\, f(j\omega) - \omega^2$ curve has its minimum at a finite frequency but let us mainly consider this case only.

(6) At this point Brune has completed one cycle. Supposing his impedance function (after separation of originally existing zeros and poles on the boundary) is of the $\frac{n^{t\lambda}}{n^{t\lambda}}$ degree in λ; then its power is first stepped up to the $\frac{(n+1)^{t\lambda}}{n^{t\lambda}}$ degree, then decreased to the $\frac{(n-1)^{t\lambda}}{(n-2)^{t\lambda}}$ degree and finally to the $\frac{(n-2)^{t\lambda}}{(n-2)^{t\lambda}}$ degree, i.e. a decrease of 2 units.

After having formed two determinants[193] we also have completed one cycle, and when starting out from a matrix whose determinant (after originally existing zeros and poles on the boundary have been removed) is of the $\frac{2n^{t\lambda}}{2n^{t\lambda}}$ degree in λ, we first step up its power to the $\frac{(2n+2)^{t\lambda}}{2n^{t\lambda}}$ degree; then we decrease it to the $\frac{(2n-1)^{t\lambda}}{2n^{t\lambda}}$ degree, and finally to the $\frac{(2n-2)^{t\lambda}}{(2n-2)^{t\lambda}}$ degree, i.e. also here a decrease with 2 units takes place.

(7) In every cycle Brune gets one negative and two positive inductive elements and the effect of his negative elements can be produced by using close coupling between two positive inductances, equivalent to his purely inductive T-structure.

We get in every cycle usually two negative reactance elements and a number (three maybe) of positive ones. Whether the effect of these negative elements can be produced in an analogous manner as in the single function case is not known, but it is very probable.

It would be in a rather complicated system of connections, however, and the matter loses its importance after having read the following Part, which provides us with another general method of realization not bringing in the question of negative elements.

[193] That is, after two inversions.

(g). It is not always, however, that the lowest point of the $\frac{r_1 r_n - r_{1n}^2}{r_n} - \omega^2$ curve, say, is falling at a finite frequency making negative reactance elements to come into play.

For functions generally corresponding to a dissipative four-terminal network containing two kinds of elements only, the lowest point of the above curve, and of the analogous curves, is found to fall at the origin or at infinity respectively, where the imaginary part of each one of the α:s and β:s is zero.

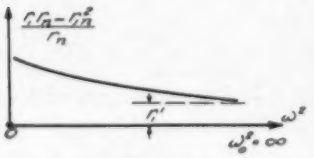


FIG. 50

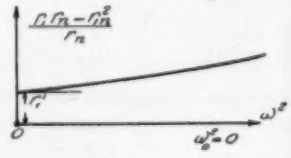


FIG. 51

Thus when considering a "positive real" α-matrix corresponding to a network having resistance and inductance elements, the lowest point of the $\frac{r_1 r_n - r_{1n}^2}{r_1} - \omega^2$ or of the $\frac{r_1 r_n - r_{1n}^2}{r_n} - \omega^2$ curve lies at ∞, as shown in Fig. 50, and when corresponding to resistance and capacitance elements it lies at the origin, as shown in Fig. 51. The reverse is true regarding the lowest point of the $\frac{h_1 h_n - h_{1n}^2}{h_1} - \omega^2$ or the $\frac{h_1 h_n - h_{1n}^2}{h_n} - \omega^2$ curve, when considering the β:s.

In this case, however, an extraction of a constant minimum value has to be made before every time we invert, i.e. (when starting out from a set of β:s say) besides extracting h_1' corresponding to the lowest point on the $\frac{h_1 h_n - h_{1n}^2}{h_n} - \omega^2$ curve we also have to extract an analogous r_1'-value after one inversion.

Note, however, that the lowest point of each one of the separate

$r_1 - \omega^2$ and $r_n - \omega^2$ curves, or of the $h_1 - \omega^2$ and $h_n - \omega^2$ curves, for cases considered in this item falls at the same limiting frequency as that at which the lowest point of the above corresponding curves fall, and it seems that when this happens we can always extract these "minimum" values together with the corresponding value from the real part of the transfer function, making the separate r:s or h:s, as well as their determinants, have a zero at the same point in question.

For an illustration see *Example* 24.[194]

Consequently from the content of this item we see that for the realization of a "positive real" matrix, generally corresponding to a dissipative four-terminal network having two kinds of elements, the general method outlined in this Part is perfectly good, and we could state another Theorem which, however, would be a repetition of Theorem XIII, the truth of which herewith has been endorsed.

[194] Appendix, p. 239.

PART XI

A GENERAL METHOD OF FINDING A NETWORK REPRESENTATION OF ANY "POSITIVE REAL" α- OR β-MATRIX.

In this final Part we shall outline a general method of realization which completely solves our problem of synthesizing a finite four-terminal network; the purely reactive case, solved in Part VIII, being included in this method.

(a). The general method outlined in the following supersedes and replaces the one outlined in Part X, at least when the prescribed "positive real" matrix refers to a four-terminal network having all three kinds of elements.

It is based upon the following principles,[195] several of them already dealt with in the foregoing:

(1) Every "positive real" α- or β-matrix, whose complex determinant has (besides in the interior of the left half of the λ-plane) zeros or (and) poles on the boundary, can always be reduced to a "positive real" α- or β-matrix whose determinant does not contain such zeros or (and) poles.

Further, poles on the boundary removed from the elements of the prescribed matrix, or (and) from the elements of its inverse matrix, can always separately be realized into purely reactive component structures to be joined to the rest of the network representation as shown in the foregoing.

(2) Every "positive real" matrix: $\left\| \begin{matrix} f_{11} & f_{1n} \\ f_{1n} & f_{nn} \end{matrix} \right\|$ can be re-

[195] Each point will be considered in more detail later in this Part.

solved[196] into two "positive real" matrices: $\left\| \begin{matrix} f_{11} & f_{1n} \\ f_{1n} & f'_{nn} \end{matrix} \right\| + \left\| \begin{matrix} 0 & 0 \\ 0 & f_n \end{matrix} \right\|$; f_n being generated from its real part and such that the first matrix has the property that $Ref_{11}\ Ref'_{nn} - [Ref_{1n}]^2 =$ zero along the whole boundary. The first matrix is called the "*reduced*" matrix, and the second one the "*extracted*" matrix, and *the same attributes are given to the corresponding networks.*

(3) It is known[197] that *the real part* of each element of a "positive real" matrix *is an even function of* ω, *while the imaginary part is an odd function of* ω. Thus each one of our three functions, when referring to the boundary, may be written:

$$f_{(j\omega)} = H_{(\omega^2)} + j\omega J_{(\omega^2)} \qquad \ldots\ldots (180)$$

(4) That fact is of utmost importance, because then, *after point (1) has been considered, the imaginary part of each* α- *and* β-*function is always zero at the origin and at* ∞. Thus we see from (166) and (167) that the determinant of the "reduced" matrix invariably has two (at least) zeros on the boundary, one at the origin and one at ∞, making it possible to remove corresponding poles from the elements of the inverse matrix, as seen from equations (15) and (16).

Then the same feature shows up again and again, and for each time we invert, generally two poles on the boundary can be removed from each function.

(5) In the meantime *invaluable cancellations of common factors take place*; the denominator of the prescribed functions becoming a factor in the numerator of the determinant of the "reduced" matrix, and then analogous cancellations occur for every time

[196] Another equivalent way of resolving said matrix would be:

$$\left\| \begin{matrix} f_{11} & f_{1n} \\ f_{1n} & f_{nn} \end{matrix} \right\| = \left\| \begin{matrix} f_1 & 0 \\ 0 & 0 \end{matrix} \right\| + \left\| \begin{matrix} f'_{11} & f_{1n} \\ f_{1n} & f_{nn} \end{matrix} \right\|$$

in which case the first one of the matrices to the right is the "extracted" matrix and the second one is the "reduced" matrix. Should the determinant of the real parts be singular originally, then no splitting should be done.

[197] Bibliogr. 38, p. 63.

we invert from α- to β-functions or vice versa, regardless of the subsequent removal of poles on the boundary.

(6) Further, the property of the "reduced" matrix, that the determinant of the real parts of its elements is zero, *holds for every following matrix* obtained by an inversion according to (15) and (16). Then by repeated inversions, and removals of poles at the origin and at ∞ in the meantime, the "reduced" matrix *is ultimately reduced to a matrix of constants or eventually to a singular matrix having complex elements.*

(7) *In addition a simple method of generating the* λ*-function in the "extracted" matrix from an* ω*-function, supposed to be its real part along the boundary, is known,*[198] and said λ-function, arrived at in this manner, can be proved *always to be a "positive real" function* realizable into a dissipative, finite, passive two-terminal network to be connected to the rest of the network representation according to the character of the prescribed matrix.

Then the driving-point function of the reduced network, at the end where a dissipative two-terminal network has been bridged across its main terminals (if considering a prescribed "positive real" α-matrix) or where a dissipative two-terminal network has been connected in series with one of the entering main leads (if considering a prescribed "positive real" β-matrix), is simply the difference between the originally prescribed driving-point function (having originally existing poles on the boundary removed) at this end, and the generated "positive real" function discussed above. Said driving-point function can, as a check, also readily be generated from its real part along the boundary.

Let us now consider each point in more detail:

(**b**). As to point (1), it is clear that because the prescribed functions are elements of a "positive real" matrix, among others, items (i), (ii) and (iii) under the definition of such a

[198] See item (b), Part IX.

matrix are fulfilled. Thus every component family of functions, arising from poles on the boundary and separated by means of a partial fraction expansion, *is always realizable into a passive purely reactive structure*, and because any two members of such a family form a constant ratio the method of design is known from the foregoing.

If the determinant of the prescribed matrix has zeros on the boundary, then we may or may not invert and "remove the zeros," as in most cases the total network representation becomes more complicated if an inversion is done in order to remove such zeros in the form of poles of the elements of the inverse matrix.

(c). As to the splitting procedure, dealt with in point (2), the matrix equation, when considering a prescribed "positive real" α-matrix, would be:

$$\begin{Vmatrix} \alpha_{11} & \alpha_{1n} \\ \alpha_{1n} & \alpha_{nn} \end{Vmatrix} = \begin{Vmatrix} Y_1 & 0 \\ 0 & 0 \end{Vmatrix} + \begin{Vmatrix} \alpha'_{11} & \alpha_{1n} \\ \alpha_{1n} & \alpha_{nn} \end{Vmatrix} = \begin{Vmatrix} \alpha_{11} & \alpha_{1n} \\ \alpha_{1n} & \alpha'_{nn} \end{Vmatrix} + \begin{Vmatrix} 0 & 0 \\ 0 & Y_n \end{Vmatrix} \quad \ldots (181)$$

$$\leftarrow \text{Alternative I} \rightarrow \qquad \leftarrow \text{Alternative II} \rightarrow$$

where Alternative I refers to the case when α_{11} has been resolved into two components, α'_{11} and Y_1, corresponding to bridging a finite two-terminal network (the "extracted" network), having Y_1 as its driving-point admittance, across the left main terminals of the total network representation of the prescribed α-matrix. Alternative II, represented in Fig. 52 (including the network representation of originally existing poles on the boundary) below, is the corresponding case when resolving α_{nn} into two components α'_{nn} and Y_n.

As seen from (181), only one element of the prescribed α-matrix is changed in each alternative, respectively.

Considering Alternative II say, and solving for α'_{nn} being a driving-point admittance of the "reduced" network, we get:

$$\alpha'_{nn} = \alpha_{nn} - Y_n \quad \ldots\ldots (182)$$

By the aid of (47) and letting:

$$\left.\begin{aligned} \alpha'_{nn} &= r'_n + jx'_n \\ Y_n &= Y_{nr} + jY_{nx} \end{aligned}\right\}$$

then, by (182), we have:

$$r'_n = r_n - Y_{nr} \qquad \ldots\ldots (182a)$$

Y_{nr} has now to be adjusted in such a manner that the determinant $(r_1 r'_n - r^2_{1n})$ becomes $= 0$, i.e. when solving for r'_n in this

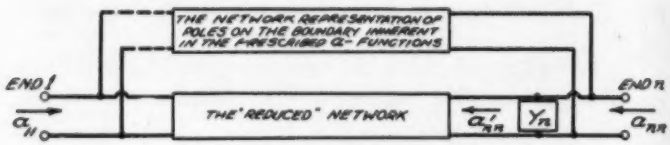


FIG. 52

determinant equation and alternatively when substituting the expression for r'_n of (182a) into this determinant and then solving for Y_{nr} we get the equations:

$$\left.\begin{aligned} Y_{nr} &= \frac{r_1 r_n - r^2_{1n}}{r_1} \\ r'_n &= \frac{r^2_{1n}}{r_1} \end{aligned}\right\} \qquad \ldots\ldots (183)$$

The functions of (183) may refer to the whole right half of the λ-plane i.e. they are generally potential functions of both γ and ω.

We shall now have to show that Y_{nr} and r'_n can necessarily each be the real part of a "positive real" function!

In order to do that, let us proceed as follows:

(A) First we shall show that the λ-functions to be generated by Y_{nr} and r_n' are regular in the right half of the λ-plane, including the imaginary axis, i.e. no poles exist there. Note, this then permits us to limit our efforts to the boundary, but it does

not tell us anything about the question whether Y_n and α'_{nn} are "positive real" functions or not, because, for instance, *our transfer function always fulfils the regularity clause, but on the other hand it may or may not be a "positive real" function.*

(B) Then we shall show that Y_{nr} and r'_n are each $\geq$ zero, for $\gamma \geq$ zero.

As to point (A) we know that *poles on the boundary* present in our α- or β-functions *do not appear in the real part on the boundary* but zeros do.

If Y_n or α_{nn} should have a pole to the right of the boundary, then the real, as well as the imaginary, part would become infinite somewhere infinitesimally close to this pole, when approaching it in a certain direction.

A pole of Y_n or α'_{nn} located to the right of the boundary in the λ-plane would, however, necessarily have been produced by a root of the integral λ-function generated by the numerator of r_1.

Such a root cannot appear, however, and even if it could its corresponding factor in the denominator of Y_{nr} and r_n' in (183) would cancel out against the same factor in the numerator of said functions, due to (60).[199]

Consequently the λ-functions to be generated by (183) *fulfil Cauchy's line integral requirements in the right half of the λ-plane, and, like the case when dealing with our ordinary α:s or β:s, we can limit our efforts to the boundary.*

As to point (B) we see from (183) and from the positiveness conditions of Part III that Y_{nr} and r'_n are in addition each ≥ 0, for $\gamma \geq 0$, *i.e. they are the real parts of two "positive real" functions.*

Let us, however, assume that we only know that Y_n and α_{nn} are *regular* in the right half of the λ-plane, making it possible to consider their real parts along the boundary only, and that we then have to test out these real parts in order to show that they can be the real parts of two "positive real" functions.

[199] Page 53.

A function, in terms of ω only, must satisfy three conditions for being usable as the real part of a "positive real" function, these conditions being:

(1) It must be an even[200] function of ω.

(2) It must be $\geq$ zero, for all real ω:s.

(3) It is not allowed to become infinite for any real ω.

As to point (1), we see immediately from (183) that it is satisfied by Y_{nr} and r'_n, being quotients of real parts only, and each such real part is individually an even function of ω.

Further, due to the fact that the three prescribed functions satisfy the conditions of Theorem V, we see that also the requirement of point (2) is fulfilled by Y_{nr} and r'_n.

Point (3) has to be considered in more detail, and it may be pointed out that a potential function, considered along the boundary only, may become infinite for two reasons:

(3a) The power of ω in the numerator may be of higher degree than in the denominator.

(3b) Roots falling on the boundary may be contained in the denominator.

Regarding (3a) we know that we are safe as to r_1 and r_n, each being the real part of a "positive real" function. Further it cannot happen to r_{1n} due to condition (60). Then we can show that we also are safe as to Y_{nr} and r'_n in (183), because:

$$\left.\begin{array}{l} \text{Suppose } r_1 \text{ is of the } \dfrac{(k-m)^{th}}{k^{th}} \text{ degree in } \omega \\ \text{“ } r_n \text{ “ “ “ } \dfrac{(k-n)^{th}}{k^{th}} \text{ “ “ “} \\ \text{“ } r_{1n} \text{ “ “ “ } \dfrac{u^{th}}{k^{th}} \text{ “ “ “} \end{array}\right\}$$

We know that r_{1n}^2 is $\leq r_1 r_n$

[200] This necessity holds for the real part of any rational fractional function of λ, whether it is a "positive real" one or not.

Consequently, we also know that:[201]

$$2u \leq 2k - m - n$$

From the expression $\frac{r_{1n}^2}{r_1}$ we obtain something of the $\frac{2u^{th}}{(2k-m)^{th}}$ degree in ω.

To show that $2u$ *is* $\leq 2k - m$!

This is simple, because we know that $2u$ is $\leq 2k - m - n$.

Thus, due to the fact that m and n are positive constants or zero, $2u$ *is* necessarily $\leq 2k - m$, and r_n' of (183) is safely guarded against (3a).

It remains to investigate Y_{nr} in respect to the same point.

Y_{nr}, may however, be written:

$$Y_{nr} = r_n - \frac{r_{1n}^2}{r_1}.$$

We now know that not only r_n but also $\frac{r_{1n}^2}{r_1}$ is safely guarded as to (3a). By multiplying and dividing the first term by r_1, and thus forming $\frac{r_1 r_n - r_{1n}^2}{r_1}$, we do not change anything, and consequently also as to Y_{nr} we are safe.

Then we proceed to consider (3b).

We know that any one of our functions can have a zero on the boundary and that such a zero will appear in both the real and the imaginary part.

In (183) the real parts only are included, however, and in case r_1 (or r_n) should happen to contain a zero on the boundary then r_{1n}, due to condition (60), must contain a zero for the same value of ω, and *consequently a zero of* r_1 *on the boundary always cancels out in both expressions of (183) safe-guarding* Y_{nr} *and* r'_n *also as to point (3b).*

Thus *again we have shown that* Y_{nr} *and* r_n' *can each be the real part of a "positive real" function!*

[201] Note, all letters contained are exponential values.

It may be pointed out that in case (183) had contained expressions like $\frac{r_1^2}{r_n}$ or $\frac{r_n^2}{r_1}$ we would have been very unfortunate, and our whole general method would have broken down.

It is true that both these ratios are individually even functions of ω and $\geq$ zero for all real ω:s, but this is not enough, as can be seen from *Example* 25.[202]

In respect to (3a) an expression $\frac{r_1^2}{r_n}$ would require $n > 2m$, while an expression $\frac{r_n^2}{r_1}$ would require $m > 2n$, which is not always true. Further, in respect to (3b) a pole of $\frac{r_1^2}{r_n}$ or $\frac{r_n^2}{r_1}$ on the boundary would not always cancel out against a zero at the same point.

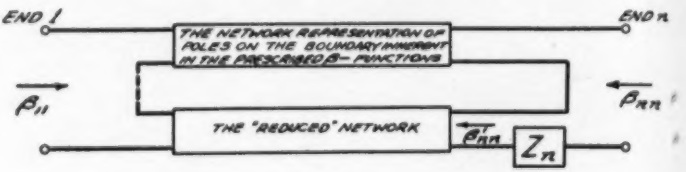


FIG. 53

(d). For the sake of completeness we consider in this item the splitting procedure of a prescribed "positive real" β-matrix.

In this case the matrix equation to consider would be:

$$\left\| \begin{matrix} \beta_{11} & \beta_{1n} \\ \beta_{1n} & \beta_{nn} \end{matrix} \right\|' = \left\| \begin{matrix} Z_1 & 0 \\ 0 & 0 \end{matrix} \right\| + \left\| \begin{matrix} \beta'_{11} & \beta_{1n} \\ \beta_{1n} & \beta_{nn} \end{matrix} \right\| = \left\| \begin{matrix} \beta_{11} & \beta_{1n} \\ \beta_{1n} & \beta'_{nn} \end{matrix} \right\| + \left\| \begin{matrix} 0 & 0 \\ 0 & Z_n \end{matrix} \right\| \quad . . (184)$$

$\leftarrow$Alternative I$\rightarrow$ $\leftarrow$Alternative II$\rightarrow$

where alternative I refers to the case when β_{11} has been resolved into two components, β'_{11} and Z_1, and alternative II to the case when β_{nn} has been resolved into two components, β'_{nn} and Z_n.

[202] Appendix, p. 243.

If reckoning with inherent poles on the boundary, then the total network representation, in the case of alternative II, of a prescribed β-matrix is indicated in Fig. 53.
Again notice that also in the case of a prescribed β-matrix, only one element is changed in each alternative.

Solving for the driving-point impedance β'_{nn} of the "reduced" network, referring to alternative II, we get:

$$\beta'_{nn} = \beta_{nn} - Z_n \qquad \ldots\ldots(185)$$

Then when letting:

$$\left.\begin{aligned} \beta'_{nn} &= h'_n + jv'_n \\ Z_n &= Z_{nr} + jZ_{ns} \end{aligned}\right\}$$

we get, by the aid of (62), after a similar procedure as shown in the foregoing:

$$\left.\begin{aligned} Z_{nr} &= \frac{h_1 h_n - h_{1n}^2}{h_1} \\ h'_n &= \frac{h_{1n}^2}{h_1} \end{aligned}\right\} \qquad \ldots\ldots(186)$$

Each one of the functions Z_{nr} and h'_n can now be proved to be the real part of a "positive real" function (Z_n and β'_{nn} respectively), and a similar discussion as given under the preceding case applies here.

(e). As to point (3), it explains itself and thus we proceed to point (4).

Considering the α-determinant say, we see from (166), (rewritten below):

$$(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2)_{(j\omega)} = (r_1 r_n - r_{1n}^2) - (x_1 x_n - x_{1n}^2) + j(r_1 x_n + r_n x_1 - 2r_{1n}x_{1n}) \qquad .(166)$$

that because now the r-determinant is equal to zero for all ω:s and because originally existing poles on the boundary have been

removed, none of the x:s can have a pole at the origin or at ∞. In addition, when knowing that the highest power of ω^2 in the numerator of each one of the r:s always is $\leq$ the one in their denominator, and that the x:s are odd functions of ω, i.e. each imaginary part has an ω as a factor in its numerator, then it is clear that *the α-determinant of* (166) *has* (at least) *two zeros on the boundary, one at the origin and one at ∞, making poles at these points to generally appear in the elements of the β-matrix, and vice versa if considering the β-determinant of* (167).

As to the zero at infinity, we know that originally existing poles there have been removed. Then it becomes clear that the x:s and v:s of (166) and (167) respectively, if having ω to the n^{th} power in the denominator, they then have ω to at most the $(n-1)^{th}$ power in the numerator, because they are odd functions of ω.

Thus, due to this feature and besides due to the limitations as to the power of ω^2 in the numerator and the denominator of the r:s and h:s, we see that for $\omega = \infty$ the α-determinant and β-determinant respectively again become equal to zero.

The fact that for each time we invert (after removing poles), the determinant of a set of functions gets one zero at the origin and at ∞, naturally means that if say:

$$\left.\begin{aligned} \alpha_{11}\alpha_{nn} &= \frac{a_r\lambda^r + a_{r-1}\lambda^{r-1} + \ldots + a_1\lambda + a_0}{h_{(\lambda)}{}^2} \\ \text{and} \qquad \alpha_{1n}^2 &= \frac{b_r\lambda^r + b_{r-1}\lambda^{r-1} + \ldots + b_1\lambda + b_0}{h_{(\lambda)}{}^2} \end{aligned}\right\}$$

then a_r is always equal to b_r, and a_0 is always equal to b_0, i.e. in each inversion these terms always cancel out.

The statement "at least" to be found under point (4) refers to the rather unlikely possibility of (166) or (167) having a zero at a *finite* ω.

When this happens, then two (conjugate) additional poles can

generally be removed from each member of the next family, decreasing the power of λ in the numerator as well as in the denominator by two additional units, although the removal of such poles is not necessary.

Such an occurrence does not introduce any "negative" reactance elements, however, because such elements come only into play when we *make* (166) or (167) zero at a finite frequency by extracting real quantities, and introducing poles at ∞ or at the origin.

For (166) to become zero at a finite ω, we can deduce a simple relation which then must hold:

From the first two parts substitute $r_1 = \frac{r_{1n}^2}{r_n}$ and $x_1 = \frac{x_{1n}^2}{x_n}$ into the last part, equated to zero, giving:

$$\frac{r_{1n}^2}{r_n} x_n + r_n \frac{x_{1n}^2}{x_n} - 2 r_{1n} x_{1n} = 0$$

which simplifies into:

$$(r_{1n} x_n - r_n x_{1n})^2 = 0$$

i.e. then for some finite real ω:

$$r_{1n} x_n \textit{ must be } = r_n x_{1n} \qquad \ldots\ldots (187)$$

Several other relations can be obtained, by selecting other substitutions, but they are all equivalent, and if (187) is fulfilled then they would all be fulfilled.

(f). As to point (5) let us divide up the item below into two sub-items as follows:

(1) *Suppose the determinant of the real parts of our functions is originally equal to zero.*

To prove that the denominator of the prescribed functions becomes a factor in the numerator of their determinant!

Let the functions (α:s say) be those of (1), rewritten below:

$$\left.\begin{aligned} \alpha_{11} &= \frac{g_{1(\lambda)}}{h_{(\lambda)}} \\ \alpha_{nn} &= \frac{g_{n(\lambda)}}{h_{(\lambda)}} \\ \alpha_{1n} &= \frac{g_{1n(\lambda)}}{h_{(\lambda)}} \end{aligned}\right\} \quad \ldots\ldots(1)$$

The numerators and denominators are generally complex, and thus (1) may be written:

$$\left.\begin{aligned} \alpha_{11} &= \frac{A_1 + jA_1'}{B + jB'} = r_1 + jx_1 \\ \alpha_{nn} &= \frac{A_n + jA_n'}{B + jB'} = r_n + jx_n \\ \alpha_{1n} &= \frac{A_{1n} + jA_{1n}'}{B + jB'} = r_{1n} + jx_{1n} \end{aligned}\right\} \quad \ldots\ldots(188)$$

For the sake of generality all quantities of (188) may be considered as functions of both[203] γ and ω.

The r:s of (188) are found to be:

$$\left.\begin{aligned} r_1 &= \frac{A_1B + A_1'B'}{B^2 + B'^2} \\ r_n &= \frac{A_nB + A_n'B'}{B^2 + B'^2} \\ r_{1n} &= \frac{A_{1n}B + A_{1n}'B'}{B^2 + B'^2} \end{aligned}\right\} \quad \ldots\ldots(189)$$

and the r-determinant equated to zero is:

$$(r_1r_n - r_{1n}^2) = \frac{(A_1A_n - A_{1n}^2)B^2 + (A_1A_n' + A_nA_1' - 2A_{1n}A_{1n}')BB' + (A_1'A_n' - A_{1n}'^2)B'^2}{(B^2 + B'^2)^2} = 0 \ldots(190)$$

[203] Or of ω only!

On the other hand, when forming the α-determinant by the aid of (188), we get:

$$(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) = \frac{(A_1A_n - A_{1n}^2) - (A_1'A_n' - A_{1n}'^2) + j(A_1A_n' + A_nA_1' - 2A_{1n}A_{1n}')}{(B + jB')^2} \quad \ldots.(191)$$

But from (190)

$$(A_1A_n' + A_nA_1' - 2A_{1n}A_{1n}') = \frac{-(A_1A_n - A_{1n}^2)B^2 - (A_1'A_n' - A_{1n}'^2)B'^2}{BB'} \quad \ldots.(192)$$

Substitute this expression into (191) and we get:

$$(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) = \frac{(A_1A_n - A_{1n}^2)BB' - (A_1'A_n' - A_{1n}'^2)BB' - j[(A_1A_n - A_{1n}^2)B^2 + (A_1'A_n' - A_{1n}'^2)B'^2]}{BB'(B + jB')^2} \ldots(193)$$

But the numerator of (193) can be factored as:

$$(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) = \frac{[-(A_1'A_n' - A_{1n}'^2)B' - j(A_1A_n - A_{1n}^2)B](B + jB')}{BB'(B + jB')^2} \ldots(194)$$

i.e. the denominator of the prescribed functions is a factor in the numerator of their determinant and cancels out!

As to the factor BB' in the denominator of (194), it must also be a factor in the numerator, because it necessarily is a factor in the numerator of (192), due to the fact that the A:s, A':s, B:s and B':s are all integral functions, and *a sum of some integral functions cannot possibly be equal to a sum of fractional functions!*

For an illustration see *Example* 26.[204]

(2) *In this sub-item we shall consider the case when the determinant of the real parts of the prescribed functions is not equal to zero for all real* ω:s *originally, but is made to be so by extracting a*

[204] Appendix, p. 244.

"positive real" function from either one of the two driving-point functions, as outlined in items (c) and (d).

In this case the denominator of the prescribed functions becomes a factor in the numerator of the determinant of the "reduced" matrix for the following reasons:

Let α_{11} and α_{1n} remain unchanged and as given in (1). Further let:

$$\alpha'_{nn} = \frac{g'_{n(\lambda)}}{h_{(\lambda)}h'_{(\lambda)}}$$

Then multiplying the numerator and the denominator of α_{11} and α_{1n} with $h'_{(\lambda)}$ and using our proof in the preceding subitem, then we get:

$$\frac{g_{1(\lambda)}h'_{(\lambda)}}{h_{(\lambda)}h'_{(\lambda)}} \times \frac{g'_{n(\lambda)}}{h_{(\lambda)}h'_{(\lambda)}} - \frac{g^2_{1n(\lambda)}h'^2_{(\lambda)}}{h^2_{(\lambda)}h'^2_{(\lambda)}} = \frac{h_{(\lambda)}h'_{(\lambda)}f_{(\lambda)}}{[h_{(\lambda)}h'_{(\lambda)}]^2}^{204a}$$

This could as well have been written:

$$\frac{h'_{(\lambda)}[g_{1(\lambda)}g'_{n(\lambda)} - g^2_{1n(\lambda)}h'_{(\lambda)}]}{h^2_{(\lambda)}h'^2_{(\lambda)}}$$

i.e. $[g_{1(\lambda)}g'_{n(\lambda)} - g^2_{1n(\lambda)}h'_{(\lambda)}]$, being *the numerator of the determinant of our new set of functions in case α_{11} and α_{1n} are not multiplied and divided with $h'_{(\lambda)}$, must contain $h_{(\lambda)}$ as a factor!*

Thus we can state:

THEOREM XIV:

If a "positive real" α- or β-matrix has the specific property (which always can be obtained) that the determinant of the real parts of its elements is equal to zero along the whole boundary, then the smallest common denominator of the elements of the matrix in question is always a factor in the numerator of its determinant.

[204a] Where $f_{(\lambda)}$ stands for what is left after the common denominator of the separate functions has been factored out in the numerator of the determinant.

(g). As to point (6), let us consider a set of prescribed α:s (say) and assume that poles on the boundary, if originally present, have been removed, and further that the determinant of the real parts is equal to zero (being so originally or made so by applying the procedure outlined in point (2)).

Then, when forming the β:s, we, by the aid of (47), (62), (16) and (166), get:

$$\left.\begin{aligned}
\beta_{11} = h_1 + jv_1 &= \frac{r_n + jx_n}{-(x_1x_n - x_{1n}^2) + j(r_1x_n + r_nx_1 - 2r_{1n}x_{1n})} \\
\beta_{nn} = h_n + jv_n &= \frac{r_1 + jx_1}{-(x_1x_n - x_{1n}^2) + j(r_1x_n + r_nx_1 - 2r_{1n}x_{1n})} \\
\beta_{1n} = h_{1n} + jv_{1n} &= \frac{r_{1n} + jx_{1n}}{-(x_1x_n - x_{1n}^2) + j(r_1x_n + r_nx_1 - 2r_{1n}x_{1n})}
\end{aligned}\right\} . \quad (195)$$

and solving for the real parts (the h:s), we have:

$$\left.\begin{aligned}
h_1 &= \frac{-r_n(x_1x_n - x_{1n}^2) + x_n(r_1x_n + r_nx_1 - 2r_{1n}x_{1n})}{(x_1x_n - x_{1n}^2)^2 + (r_1x_n + r_nx_1 - 2r_{1n}x_{1n})^2} \\
h_n &= \frac{-r_1(x_1x_n - x_{1n}^2) + x_1(r_1x_n + r_nx_1 - 2r_{1n}x_{1n})}{(x_1x_n - x_{1n}^2)^2 + (r_1x_n + r_nx_1 - 2r_{1n}x_{1n})^2} \\
h_{1n} &= \frac{-r_{1n}(x_1x_n - x_{1n}^2) + x_{1n}(r_1x_n + r_nx_1 - 2r_{1n}x_{1n})}{(x_1x_n - x_{1n}^2)^2 + (r_1x_n + r_nx_1 - 2r_{1n}x_{1n})^2}
\end{aligned}\right\} . . \quad (196)$$

Thus, the h-determinant is:

$$(h_1h_n - h_{1n}^2) = \frac{\begin{array}{l}(r_1r_n - r_{1n}^2)(x_1x_n - x_{1n}^2)^2 + \\ (x_1x_n - x_{1n}^2)(r_1x_n + r_nx_1 - 2r_{1n}x_{1n})^2 \\ - (x_1x_n - x_{1n}^2)(r_1x_n + r_nx_1 - 2r_{1n}x_{1n})^2\end{array}}{[(x_1x_n - x_{1n}^2)^2 + (r_1x_n + r_nx_1 - 2r_{1n}x_{1n})^2]^2} . \quad (197)^{205}$$

From (197) *it is now clear that as* $(r_1r_n - r_{1n}^2)$ *is equal to zero, then* $(h_1h_n - h_{1n}^2)$ *is also equal to zero!*

[205] The lower lines are a continuation of the upper one.

Further, had we started out from an h-determinant being equal to zero, then the r-determinant is naturally also equal to zero. This fact can also readily be seen from (64).[206] Further it is true regardless of removal of poles produced on the boundary during the inversion process, wherefore we can state:

THEOREM I, COROLL. 1:

If the determinant of the real parts of the elements of a non-singular "positive real" α- or β-matrix is equal to zero (made so or having that specific feature originally), *then the determinant of the real parts of the elements of its inverse matrix is also equal to zero, and for every time we "invert" we get a matrix having the same property, regardless of whether produced poles on the boundary have in the meantime been removed or not.*

As to point (7), see item (b) Part IX.

(h). Let us in this item illustrate our general method during one cycle. Then the following cycles are exactly analogous, *although only once we need to "extract" a "positive real" function.*

Thus let us assume that each one of a set of prescribed α:s, having no poles on the boundary, is of the $\frac{n^{th}}{n^{th}}$ degree in λ, and that the determinant of their real parts along the boundary is $>$ zero for some, or all, real ω:s.

Then the determinant of said α:s is of the $\frac{2n^{th}}{2n^{th}}$ degree in λ.

After having "split up" α_{nn}, say, the "positive real" function Y_n, being an element of the "extracted" matrix, is readily generated by its real part and separately realized into a finite, dissipative two-terminal network.

The generated Y_n as well as $\alpha'_{nn} = \frac{g'_{n(\lambda)}}{h_{(\lambda)}h'_{(\lambda)}}$ is of the $\frac{2n^{th}}{2n^{th}}$ degree in λ.

[206] Page 58.

The elements of the "reduced" matrix are:

$$\left.\begin{aligned} \alpha_{11} &= \frac{g_{1(\lambda)}}{h_{(\lambda)}} \\ \alpha'_{nn} &= \frac{g'_{n(\lambda)}}{h_{(\lambda)}h'_{(\lambda)}} \\ \alpha_{1n} &= \frac{g_{1n(\lambda)}}{h_{(\lambda)}} \end{aligned}\right\} \qquad \ldots\ldots(198)$$

Their determinant:

$$(\alpha_{11}\alpha'_{nn} - \alpha^2_{1n}) = \frac{\lambda h_{(\lambda)}\Delta_{(\lambda)}}{h^2_{(\lambda)}h'_{(\lambda)}}$$

which ordinarily would have been of the $\frac{3n^{th}}{3n^{th}}$ degree, is, due to its zeros at the origin and at ∞, only of the $\frac{(3n-1)^{th}}{3n^{th}}$ degree, with a λ as a factor in the numerator. In addition we know that $h_{(\lambda)}$ is a factor in the numerator, i.e. after cancellation of this latter factor the above determinant is of the $\frac{(2n-1)^{th}}{2n^{th}}$ degree in λ only.

Now we invert to obtain the β:s, and by the aid of (16) we get:

$$\left.\begin{aligned} \beta_{11} &= \frac{g'_{n(\lambda)}}{\lambda\Delta_{(\lambda)}} = \frac{k_1}{\lambda} + k'_1\lambda + \frac{g''_{1(\lambda)}}{\Delta_{(\lambda)}} \\ \beta_{nn} &= \frac{g_{1(\lambda)}h'_{(\lambda)}}{\lambda\Delta_{(\lambda)}} = \frac{k_n}{\lambda} + k'_n\lambda + \frac{g''_{n(\lambda)}}{\Delta_{(\lambda)}} \\ \beta_{1n} &= \frac{g_{1n(\lambda)}h'_{(\lambda)}}{\lambda\Delta_{(\lambda)}} = \frac{k_{1n}}{\lambda} + k'_{1n}\lambda + \frac{g''_{1n(\lambda)}}{\Delta_{(\lambda)}} \end{aligned}\right\} \ldots\ldots(199)$$

where the k:s and k':s are the residues at the origin and at ∞, respectively.

At this point it should be pointed out that, *due to the property of the functions characterizing the "reduced" network, the two sets of residues in* (199) *must each form a singular determinant.*

This is so because all the β:s have the same denominator, and we know that the k:s are nothing but the constant term in the numerator of the β:s divided by the constant term in the denominator, while the k':s are nothing but the coefficient of the highest power of λ in the numerator of the β:s divided by the corresponding coefficient in the denominator.

From our discussion on page 176, however, we see that the determinant of each set of the above mentioned terms in the numerators is zero, and consequently the determinant of each set of residues must also be zero.

In addition, *for every time we invert*, the residues in each component family of extracted poles on the boundary form a singular determinant, and as we know that the determinant of the real parts of the elements in the "reduced" matrix (and its inverses) is singular, i.e. also the ultimate determinant of the constants, then it is clear from the foregoing *that each component network contained in the "reduced" network is going to contain a single element only, and all these elements, except the one contained in the last component network, being a resistance, are purely reactive.*

Before the poles are removed the β:s in (199) are of the $\frac{2n^{th}}{(2n-1)^{th}}$ degree in λ.

Thus the members of the component family to the right (β'':s say) are of the $\frac{(2n-2)^{th}}{(2n-2)^{th}}$ degree.

Their determinant:

$$(\beta''_{11}\beta''_{nn} - \beta''^{2}_{1n}) = \frac{\lambda\Delta_{(\lambda)}\Delta_{1(\lambda)}}{\Delta^{2}_{(\lambda)}}$$

instead of being of the $\frac{(4n-4)^{th}}{(4n-4)^{th}}$ degree is, due to the zeros at

the origin and at ∞ and due to the cancellation of $\Delta_{(\lambda)}$, only of the $\frac{(2n-3)^{th}}{(2n-2)^{th}}$ degree, with a λ as a factor in the numerator.

Forming the α'':s (from the β'':s), by the aid of (15), we get:

$$\left.\begin{aligned} \alpha''_{11} &= \frac{g''_{n(\lambda)}}{\lambda\Delta_{1(\lambda)}} = \frac{k''_1}{\lambda} + k'''_1\lambda + \frac{g'''_{1(\lambda)}}{\Delta_{1(\lambda)}} \\ \alpha''_{nn} &= \frac{g''_{1(\lambda)}}{\lambda\Delta_{1(\lambda)}} = \frac{k''_n}{\lambda} + k'''_n\lambda + \frac{g'''_{n(\lambda)}}{\Delta_{1(\lambda)}} \\ \alpha''_{1n} &= \frac{g''_{1n(\lambda)}}{\lambda\Delta_{1(\lambda)}} = \frac{k''_{1n}}{\lambda} + k'''_{1n}\lambda + \frac{g'''_{1n(\lambda)}}{\Delta_{1(\lambda)}} \end{aligned}\right\} \ldots\ldots (200)$$

where the k'':s and k''':s are residues.

The α'':s are before the removal of the poles of the $\frac{(2n-2)^{th}}{(2n-3)^{th}}$ degree, i.e. the members of the component family to the right (α''':s say) are of the $\frac{(2n-4)^{th}}{(2n-4)^{th}}$ degree.

Their determinant:

$$(\alpha'''_{11}\alpha'''_{nn} - \alpha'''^2_{1n}) = \frac{\lambda\Delta_{1(\lambda)}\Delta_{2(\lambda)}}{\Delta_{1(\lambda)}^2}$$

is, after cancellations of factors, of the $\frac{(2n-5)^{th}}{(2n-4)^{th}}$ degree, with a λ as a factor in the numerator.

Here we have completed one cycle!

Note, the determinant of the prescribed "positive real" matrix was of the $\frac{2n^{th}}{2n^{th}}$ degree in λ, while above α'''-determinant is of the $\frac{(2n-5)^{th}}{(2n-4)^{th}}$ degree only, *i.e. in one single cycle the power of* λ *in the numerator has decreased 5 units and in the denominator 4 units*, in spite of the fact that the power of λ in α'_{nn} was increased (some-

what artificially) to twice as high a value as the power of λ in α_{nn}. *Then in each cycle the power of* λ *goes down 4 units, in the numerator as well as in the denominator.*

Considering the network representation of a set of prescribed α-functions (say), we see from (199) that after Y_n is extracted and separately realized we have next a sum of three β-component networks to be connected as shown in Fig. 44.

Then from (200) we see that the last one (say) of said β-component networks consists of three α-component networks. Then the last one (say) of said α-component networks consists of three β-component networks etc., until in the last group the last component network, being the realization of a matrix of constants, contains resistance only.

In the last group, naturally, less than three component networks may be present, and in any cycle perhaps a pole at the origin or at ∞ can be removed from only one of our functions (in such a case always from one of our driving-point functions).[207]

The case of removing a pole at the origin or at ∞ *from the two driving-point functions cannot happen, because,* regardless of the realization possibility of such poles, *the determinant of the set of residues in question would then not be zero.*

It should be added, that the "reduced" matrix may come out singular. Or it may happen, that at any stage of the resolution procedure the unrealized component matrix is singular. In both cases no (further) inversion can be done, but in the next two items it will be shown that any singular matrix can directly be realized into a single two-terminal network plus a transformer of known ratio.

For an illustration of the general realization method outlined in this Part see *Example* 27.[208]

(i). If a prescribed "positive real" α- or β-matrix is such that the determinant of the real parts of its elements is $\not\equiv$ zero for

[207] For this to happen two members of the preceding family must have a zero at the corresponding extreme λ-value.

[208] Appendix, p. 245.

some or all real ω:s, then it seems evident that we should apply our general method of this Part and realize said matrix, whether it is non-singular or singular.[209]

Thus, what remains, in order to show the generality of our method, is to prove that every singular "positive real" matrix, whose elements are such that the determinant of their real parts is = zero, has a finite, passive network representation.

When an α-matrix is singular, then the General Circuit Parameter C is zero, and when a β-matrix is singular then the General Circuit Parameter B is zero, and in such cases we often have

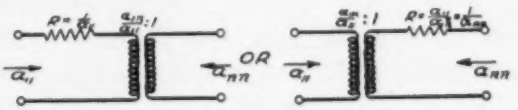


FIG. 54

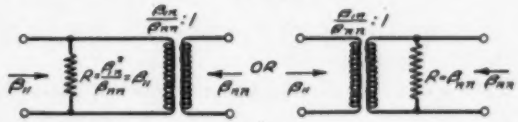


FIG. 55

noticed that the prescribed matrix has been realized in a single step, usually into a structure containing a two-terminal network and a transformer.

The ratio of this transformer must, however, naturally come out as a real constant!

Let us divide up this item into three sub-items, due to the fact that three kinds of such matrices interest us.

(1) If all the elements of the prescribed singular matrix are real constants, then the transformer ratio is naturally also constant, and it is immediately clear that such a singular α-matrix has the network representation of Fig. 54.

[209] Under the next item we shall show, however, that every singular "positive real" matrix is such that the determinant of the real (as well as of the imaginary) parts of its elements always is zero.

Further, such a singular β-matrix naturally has the network representation of Fig. 55.

For an illustration, see *Example* 28.[210]

(2) The case when the elements of a singular "positive real" matrix become purely imaginary for $\lambda = j\omega$ is then also disposed of, by noticing that the ratios of the transformers in Figs. 54 and 55 are again real constants, and consequently the above two figures show the network representation also of a singular "positive real" matrix falling under this sub-item. The resistance elements are then, however, replaced by proper reactance elements.

Note, the component families of removed poles on the boundary in our general method of this Part fall under this sub-item, because the k-determinant is always singular.

(3) Finally, we consider a singular "positive real" matrix whose elements become complex for $\lambda = j\omega$ and such that the determinant of their real parts is zero.

From Part VII we know that the transformer ratio is either $\frac{1}{D}:1$ or $A:1$, and note that in this item $\frac{1}{D}$ is $= A$, because in the relation $(AD - BC) = 1$ either B or C is zero.

For a singular "positive real" α-matrix, say, we have:

$$(\alpha_{11}\ \alpha_{nn} - \alpha_{1n}^2) = 0 \qquad \ldots\ldots(201)$$

which can be written:

$$\frac{\alpha_{nn}}{\alpha_{1n}} = \frac{\alpha_{1n}}{\alpha_{11}}$$

These ratios appear in (22), and thus we have:

$$\frac{\alpha_{nn}}{\alpha_{1n}} = \frac{\alpha_{1n}}{\alpha_{11}} = A = \frac{1}{D} \qquad \ldots\ldots(202)$$

[210] Appendix, p. 252.

To prove that the ratio in (202) always is a real constant!

The case of α_{1n} being zero can be disregarded, because from (201) we see that then either α_{11} or α_{nn} must also be zero, and all that we have to realize is a single "positive real" function.

Thus we assume α_{1n} to be $\neq 0$, and by the aid of (47) we have:

$$\frac{\alpha_{nn}}{\alpha_{1n}} = \frac{r_n + jx_n}{r_{1n} + jx_{1n}} = \frac{r_n r_{1n} + x_n x_{1n}}{r_{1n}^2 + x_{1n}^2} + j\frac{r_{1n}x_n - r_n x_{1n}}{r_{1n}^2 + x_{1n}^2} \quad \ldots (203)$$

Further:

$$\begin{aligned}(\alpha_{11}\,\alpha_{nn} - \alpha^2{}_{1n}) &= (r_1 r_n - r^2{}_{1n}) - (x_1 x_n - x^2{}_{1n}) + \\ & j(r_1 x_n + r_n x_1 - 2r_{1n}x_{1n}) = 0 \qquad \ldots\ldots (204)\end{aligned}$$

Then, knowing that the r-determinant is zero, we get:

$$\left.\begin{aligned}(r_1 r_n - r_{1n}^2) &= 0 \\ (x_1 x_n - x_{1n}^2) &= 0 \\ (r_1 x_n + r_n x_1 - 2r_{1n}x_{1n}) &= 0\end{aligned}\right\} \quad \ldots\ldots (205)$$

Solving equations (205) together we get:

$$(r_{1n}x_n - r_n x_{1n})^2 = 0$$

i.e. we have proved that the imaginary part in (203) is zero, and consequently the ratio in (202) is a *real* quantity.

In addition we must show, however, that said ratio is *constant*!

It can readily be seen that this actually is the case, because the real part and the imaginary part in (203) correspond to u and v respectively in (70).[211]

We have shown that the imaginary part mentioned is zero, i.e. the quantity v is therefore zero. Thus, its partial derivatives in (70) are also zero, making the partial derivatives of u in (70) to be zero. Consequently u itself, as well as its correspondence in (203), *is necessarily a constant*!

Thus when considering a singular "positive real" α-matrix.

[211] Page 64.

falling under this sub-item, its network representation is shown in Fig. 56.

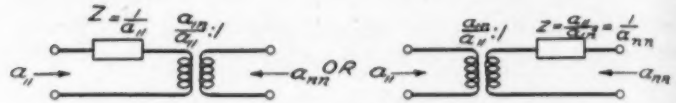


FIG. 56

and when considering such a β-matrix its network representation is shown in Fig. 57.

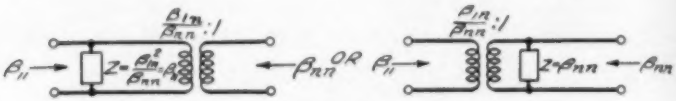


FIG. 57

For an illustration see *Example* 29.[212]

(j). In this item we shall now show that every singular "positive real" matrix is such that the determinant of the real parts of its elements is also singular, and consequently the network representations shown under the preceeding item take care of every case.

In the beginning of item (*i*) it was stated that we should apply our general method of this Part also in the case of a *singular* "positive real" matrix.

Thus, it seems as if in the case of an α-matrix its network representation should possibly have a two-terminal network bridged across one pair of the main terminals, and in the case of a β-matrix a two-terminal network should be connected in series with one of the terminating leads.

By giving more careful consideration to the case, then it becomes clear, however, that *for a singular matrix a network representation as shown in Figs. 52 and 53 respectively is absurd,*

[212] Appendix, p. 253.

because all β-functions obtained by the inversion of a singular α-matrix are infinite according to (16), and vice versa according to (15), while the β-functions of the structure in Fig. 52 and the α-functions of the structure in Fig. 53 respectively are certainly finite if the "extracted" two-terminal network is present.

Thus this network cannot in this case be allowed to appear, which means that Y_n and Z_n respectively, as well as Y_{nr} in (183) and Z_{nr} in (186) respectively, are zero, i.e. a singular "positive real" matrix has always originally the property that the determinant of the real (as well as of the imaginary) parts of its elements is equal to zero.

This fact can also be algebraically proved by the aid of the properties of a positive definite quadratic form:

$$\sum_{ij=1}^{n} a_{ij} x_i x_j \qquad \ldots\ldots (206)$$

applied upon a "positive real" matrix.

Considering such an α-matrix, say, we must, however, in the first hand show that always:

$$(r_1 + r_n \pm 2 r_{1n}) \geq 0$$

because:

$$\left.\begin{array}{ll} (r_1 r_n - r_{1n}^2) & \geq 0 \\ r_1 & \geq 0 \\ r_n & \geq 0 \end{array}\right\}$$

This is the same as to prove that:

$$\frac{r_1 + r_n}{2} \geq |\, r_{1n} \,|, \textit{ when } \sqrt{r_1 r_n} \geq |\, r_{1n} \,|$$

It is clear that:

$$(r_1 - r_n)^2 = r_1^2 + r_n^2 - 2 r_1 r_n \text{ is } \geq 0 \quad .$$

Add $4\,r_1 r_n$ on both sides and we get:

$$(r_1 + r_n)^2 \geq 4\, r_1 r_n$$

i.e.

$$\frac{r_1 + r_n}{2} \geq |\sqrt{r_1 r_n}|$$

which in words says that *the arithmetic mean is always* $\geq$ *the geometric mean*, the equality sign being valid for $r_1 = r_n$.

Thus we see that $\frac{r_1 + r_n}{2}$ certainly is $\geq |r_{1n}|$ if $\sqrt{r_1 r_n}$ is $\geq |r_{1n}|$, i.e. then:

$$r_1 + r_n \pm 2\, r_{1n} \geq 0 \qquad \ldots\ldots(207)$$

This, however, is the same as to say that the quadratic form:

$$(r_1 x_1^2 \pm 2\, r_{1n}\, x_1 x_n + r_n x_n^2) \text{ is } \geq 0 \qquad \ldots\ldots(208)$$

i.e. it is always positive definite!

Having the real parts in mind, then the quadratic form of (206) can as well be applied on the α-functions themselves, and we get:

$$(\alpha_{11} x_1^2 \pm 2\alpha_{1n}\, x_1 x_n + \alpha_{nn} x_n^2) \geq 0 \qquad \ldots\ldots(209)$$

Then, by the aid of (201), this expression goes over into:

$$\left(\alpha_{11} x_1^2 \pm 2\alpha_{1n}\, x_1 x_n + \frac{\alpha_{1n}^2}{\alpha_{11}}\, x_n\right) \geq 0$$

i.e.

$$\alpha_{11}\left(x_1 \pm \frac{\alpha_{1n}}{\alpha_{11}} x_n\right)^2 \geq 0 \qquad \ldots\ldots(210)$$

Letting:

$$\frac{\alpha_{1n}}{\alpha_{11}} = \frac{\alpha_{nn}}{\alpha_{1n}} = R + jX \qquad \ldots\ldots(211)$$

and substituting it into (210), we then have:

$$\alpha_{11}\,(x_1 \pm R\,x_n \pm jX\,x_n)^2 \geq 0 \qquad \ldots\ldots(212)$$

We know that the real part of this expression has always to be positive for *every* real x- and ω-value! On the other hand the sum $(x_1 \pm R\,x_n)$ is unquestionably zero for certain such values, in which case the positiveness requirement of the real part of (212) would be violated, because $Re\alpha_{11}$ is always ≥ 0.

Thus, X *must be zero*, i.e. the ratio in (211) is always a *real* quantity!

Then by the aid of (70) it can in addition, in the same manner as under the preceding item, be proved to be *constant*, and we can state:

THEOREM XV:

Every singular "positive real" α- or β-matrix has always the property that the determinant of the real parts and the determinant of the imaginary parts of its elements are each separately zero, and every such matrix satisfies both necessary and sufficient conditions for having a network representation as shown in Fig. 56 and Fig. 57 respectively.

(k). *With the finish of the above two items we have covered our whole problem*, and taking all the results of this paper into account it becomes evident that we have material enough for a new important Theorem, stating:

THEOREM XVI:

The term "positive real" matrix, as defined in Part V, with elements having a finite number of zeros and poles, and a matrix having as elements either the two "short-circuit driving-point" ad-

mittances and the "short-circuit transfer" admittance, or the two "open-circuit driving-point" impedances and the "open circuit transfer" impedance of a finite, passive, purely reactive or dissipative, four-terminal network are synonymous.

THUS THE CONDITIONS EMBODIED IN OUR "POSITIVE REAL" MATRIX ARE NOT ONLY NECESSARY BUT ALSO SUFFICIENT!

Q. E. D.

Conclusion

As a summary we must, with the finish of the last Part, consider our research problem completely solved.

This fact gives us an efficient tool for which many investigators have previously been looking.

Dr. *W. Cauer*[213], for instance, in one of his latest publications[214] states:

"Es sollen zu einer gegebenen zweireihigen Matrixfunktion, die einen elektrischen Vierpol als Funktion der Frequenz des Wechselstroms charakterisiert, physikalische Vierpole konstruiert und die notwendigen und hinreichenden Bedingungen für eine solche Realisierungsmöglichkeit angegeben werden. *Für die Nachrichtentechnik*, insbesondere für die elektrischen „Siebketten," *besitzt diese Frage erhebliches Interesse.*"

Further, Dr. *M. Vaulot*[215] says:[216]

"Faisons remarquer en terminant que les constructions de quadripôles données au paragraphe IV ne sont valables que pour une valeur déterminée de la fréquence. *Le problème consistant à construire un quadripôle dans lequel A, B, C, D sont des fonctions données de la fréquence est un problème beaucoup plus difficile dont nous n'avons pas la solution pour le moment.*"

Before our general methods can be applied for the design of a four-terminal network with prescribed propagation characteristic it is, however, necessary to develop a procedure for finding our three α- or β-functions from a curve representing the amplitude-frequency characteristic of the propagation function plus the two characteristic impedances. Thus the first thing we would have to do is to find an ω-function as a quotient of two integral polynomials, fitting this curve,[217] and then proceed to

[213] At the University of Göttingen, Germany.
[214] Bibliogr. 25, p. 88.
[215] Chief Engineer of the French Post, Telegraph and Telephone System (operated by the Goverment).
[216] Bibliogr. 32, p. 502.
[217] This procedure, however, is already rather well known in mathematics.

find the propagation function itself. Then said function and the two characteristic-impedences have to be transformed into a "positive real" α- or β-matrix, whereupon a network, which responds in a proper manner to an applied impulse, can be designed by the aid of our general methods.

In other words we would have to determine "die Zweireihige Matrix-Funktion" that Cauer refers to. When this "sub-problem" is solved, *then our general methods can always be applied to the design of corrective and selective networks of any kind.*

That synthesis of this kind is important is, for instance, clear from a statement of Dr. *O. J. Zobel*,[218] who in one of his papers[219] says:

"It would be most gratifying to be able to obtain directly from a desired propagation characteristic the corresponding form of network. This is generally a difficult problem and it becomes necessary to resort to simplifying methods somewhat similar to those employed in the design of electric wave-filters. One reason for this difficulty is that we are limited to physical resistance-, inductance-, and capacity elements, all of which must, in general, be positive. We would, therefore, begin with known forms of networks whose general propagation characteristics have been determined and choose from them one or more whose combination offers the possibility of giving a satisfactory desired result."

From our paper it is clear, that besides the possibility of realizing certain types[220] of "positive real" matrices we have (which is more important) in addition several general realization methods as summarized below:

(1) For functions characterizing a purely reactive four-terminal network we have *one general method*, shown in item (a), Part VIII.

(2) For functions characterizing a dissipative four-terminal network having generally two kinds of elements only we have *three general methods.*

The first one is outlined in items (b) and (c), Part VIII.

[218] At the Development and Research Department of the American Telephone and Telegraph Company.
[219] Bibliogr. 18, p. 466.
[220] See Parts VI, VII and IX.

The second one is the one of Part X with special attention to be given to item (g).

The third one is included in our general method of Part XI.

(3) For functions characterizing a four-terminal network having all three kinds of elements we have *two general methods.*

The first one is sketchily outlined in Part X and is not yet complete, due to the necessity of realizing certain combinations containing negative reactance elements, generally introduced in each cycle during the evaluation procedure.

The second one, however, outlined in Part XI, exhausts the elements of any prescribed "positive real" α- or β-matrix of their zeros and poles in a very rapid manner, and gives us a tool to attack any family of functions of any power in λ.

This is so, because generally *we need not find residues for finite frequencies,* except when realizing the extracted "positive real" function.

Further note that, after this function is generated and separately realized, *only positive elements are introduced,* and each component network of the "reduced" network contains *one single element.*

As to a *singular* "positive real" matrix it is taken care of by items (i) and (j), Part XI.

For our deductions we have usually been considering α-functions, but it may be pointed out that, due to the 100% analogy between the inherent features of a "positive real" α-matrix and a "positive real" β-matrix, *there is not a single transformation performed on a set of* α:s *which can not as well be performed on a set of* β:s, although then the network representation naturally will be different.

It is to hope that the solution of our research problem will open new development possibilities of communication transmission networks as a whole.

This especially as to the design of *amplitude-corrective* and *phase-corrective* networks (perhaps it will be possible to design

a network which can do both corrections simultaneously) as well as to the design of all kinds of *selective* networks (electric filter networks), which today play an important rôle in carrier telephony, multiplex telegraphy, radio, talking movies, pictures transmitted over telephone lines, television, etc.

Further, the solution of our problem may directly be of importance for designing certain types of relay circuits in power engineering.

APPENDIX

NUMERICAL EXAMPLES

Example 1. Consider the α:s:

$$\left.\begin{aligned} \alpha_{11} &= \frac{\lambda + 5\lambda}{2\lambda + 1} \\ \alpha_{nn} &= \frac{\lambda}{2\lambda + 1} \\ \alpha_{1n} &= \frac{2\lambda}{2\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots (1)$$

Then

$$(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) = \frac{\lambda^3 + \lambda^2}{(2\lambda + 1)^2} = \frac{\lambda^2(\lambda + 1)}{(2\lambda + 1)^2}$$

Thus, our β:s are:

$$\left.\begin{aligned} \beta_{11} &= \frac{\alpha_{nn}}{\alpha_{11}\alpha_{nn} + \alpha_{1n}^2} = \frac{2\lambda + 1}{\lambda^2 + \lambda} \\ \beta_{nn} &= \frac{\alpha_{11}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} = \frac{2\lambda^2 + 11\lambda + 5}{\lambda^2 + \lambda} \\ \beta_{1n} &= \frac{\alpha_{1n}}{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2} = \frac{4\lambda + 2}{\lambda^2 + \lambda} \end{aligned}\right\} \qquad \ldots\ldots (2)$$

i.e.

$$(\beta_{11}\beta_{nn} - \beta_{1n}^2) = \frac{(2\lambda^2 + 11\lambda + 5)(2\lambda + 1) - 4(2\lambda + 1)^2}{(\lambda^2 + \lambda)^2} = \frac{(2\lambda+1)^2}{\lambda^2(\lambda+1)}$$

and the statement about the zeros and poles of the two determinants holds.

Example 2. Suppose we have the passive four-terminal network:

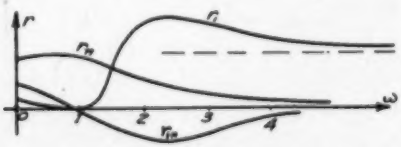
Our α:s then found to be:

$$\left.\begin{aligned} \alpha_{11} &= \frac{\lambda^3+\lambda^2+\lambda+1}{\lambda^3+\lambda^2+2\lambda} \\ \alpha_{nn} &= \frac{2\lambda^2+2\lambda}{\lambda^3+\lambda^2+2\lambda} \\ \alpha_{1n} &= \frac{-\lambda^2+\lambda}{\lambda^3+\lambda^2+2\lambda} \end{aligned}\right\} \quad \ldots\ldots(1)$$

Therefore, denoting the real parts by r_1, r_n and r_{1n} respectively, we get:

$$\left.\begin{aligned} Re\,\alpha_{11(j\omega)} &= r_1 = \frac{\omega^4-2\omega^2+1}{\omega^4-3\omega^2+4} \\ Re\,\alpha_{nn(j\omega)} &= r_n = \frac{4}{\omega^4-3\omega^2+4} \\ Re\,\alpha_{1n(j\omega)} &= r_{1n} = \frac{-2\omega^2+2}{\omega^4-3\omega^2+4} \end{aligned}\right\} \quad \ldots\ldots(2)$$

and the $(r-\omega)$ curves are:



Note, as all r:s are even functions of ω (i.e. they are functions of ω^2) the curves for negative ω:s are exactly the same as those above but turned over 180 degrees in respect to the r-axis.

Example 3. Let us consider the potential function:

$$F_{(\omega^2)} = \frac{4.48\omega^8 - 42.24\omega^6 + 139.7\omega^4 - 101.6\omega^2 + 472.3}{(4\omega^4 + 16)^2} \quad \ldots (1)$$

and see whether it is $\geq$ zero, for $0 < \omega^2 + \infty$.

The denominator of (1) can be disregarded, as always being positive, and letting the numerator be $f_{(\omega^2)}$, then we get our *Sturm functions:*

$$\left.\begin{aligned} f_{(\omega^2)} &= 4.48\omega^8 - 42.24\omega^6 + 139.7\omega^4 - 101.6\omega^2 + 472.3 \\ f'_{(\omega^2)} &= 17.92\omega^6 - 126.72\omega^4 + 279.4\omega^2 - 101.6 \\ f_{2(\omega^2)} &= 4.6\omega^4 - 87.8\omega^2 - 412.5 \\ f_{3(\omega^2)} &= -5989\omega^2 - 19200 \\ f_{4(\omega^2)} &= -74.5 \end{aligned}\right\} \ldots (2)$$

Thus we see:

	$f_{(\omega^2)}$	$f'_{(\omega^2)}$	$f_{2(\omega^2)}$	$f_{3(\omega^2)}$	$f_{4(\omega^2)}$	number of sign-variations
$\omega^2 = 0$	+	−	−	−	−	One
$\omega^2 = +\infty$	+	+	+	−	−	One

Consequently between 0 and $+\infty$ we have an equal number of sign-variations, i.e. no real roots are falling in between. As further the given curve evidently is positive for those two ω^2:s it stays above the axis all the way.

Example 4.

$$f_{(\lambda)} = \frac{5\lambda^2 + \lambda + 1}{\lambda^2 + 4\lambda + 2} \quad \ldots\ldots (1)$$

The zeros are:

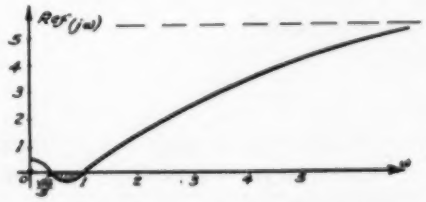
$$\left.\begin{aligned} \lambda_1 &= -\frac{1}{10} + j\frac{\sqrt{19}}{10} \\ \lambda_2 &= -\frac{1}{10} - j\frac{\sqrt{19}}{10} \end{aligned}\right\}$$

The poles are:

$$\left.\begin{aligned}\lambda_1 &= -2+\sqrt{2}\\ \lambda_2 &= -2-\sqrt{2}\end{aligned}\right\}$$

$$Re\, f_{(j\omega)} = \frac{5\omega^4 - 7\omega^2 + 2}{\omega^4 + 12\omega^2 + 4} \qquad \ldots\ldots(2)$$

i.e. zeros and poles lie to the left of the boundary, but $Re\, f_{(j\omega)}$ is negative between $\omega = \pm\frac{\sqrt{10}}{5}$ and $\omega = \pm 1$, as shown for positive ω:s by the curve below.



This curve naturally does not change if the zeros and poles of $f_{(\lambda)}$ are shifted from the left side to the right side of the boundary by a substitution: $\lambda = -\lambda$.

Example 5. Consider the "α-functions":

$$\left.\begin{aligned}\alpha_{11} &= \frac{3\lambda}{2\lambda+1}\\ \alpha_{nn} &= \frac{3\lambda}{2\lambda+1}\\ \alpha_{1n} &= \frac{\lambda^2+2\lambda}{2\lambda+1}\left(=\frac{\lambda}{2}+\frac{3/2\,\lambda}{2\lambda+1}\right)\end{aligned}\right\} \qquad \ldots\ldots(1)$$

$$\left.\begin{aligned}Re\,\alpha_{11(j\omega)} &= Re\,\alpha_{nn(j\omega)} = \frac{6\omega^2}{4\omega^2+1}\\ Re\,\alpha_{1n(j\omega)} &= \frac{3\omega^2}{4\omega^2+1}\end{aligned}\right\} \qquad \ldots\ldots(2)$$

Thus on the boundary: $r_1 \geq 0, r_n \geq 0$ and $(r_1 r_n - r_{1n}^2) \geq 0$.

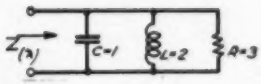
Further, all α:s are real, for λ real, and the location of zeros and poles is correct, *but note, the determinant of the residues to be* ≥ 0 *is not fulfilled*, as α_{1n} only has a pole at ∞, which is absurd.

By substituting $j\omega$ for λ in α_{1n}, before and after the pole $\frac{\lambda}{2}$ is removed, we get:

$$\frac{-\omega^2 + 2j\omega}{2j\omega + 1} = \frac{3\omega^2}{4\omega^2 + 1} + j\frac{(\ )}{(\ )}; \text{and:} \quad \frac{3/2\, j\omega}{2j\omega + 1} = \frac{3\omega^2}{4\omega^2 + 1} + j\frac{(\ )}{(\ )}$$

respectively, i.e. the *same* expression is obtained.

Example 6. Suppose we have the passive two-terminal network:



The impedance function $Z_{(\lambda)}$ is:

$$Z_{(\lambda)} = \frac{6\lambda}{6\lambda^2 + 2\lambda + 3}$$

i.e. it has one zero at ∞ and one at the origin.

Note:

$$\left[\frac{d}{d\lambda} Z_{(\lambda)}\right]_{\lambda=\infty} = \left[\frac{6(3 - 6\lambda^2)}{(6\lambda^2 + 2\lambda + 3)^2}\right]_{\lambda=\infty} \text{ is } = 0$$

Substitute $\lambda = \frac{1}{\lambda'}$ and investigate at the origin.

Thus:

$$Z_{\left(\frac{1}{\lambda'}\right)} = \frac{6\lambda'}{3\lambda'^2 + 2\lambda' + 6}$$

and

$$\left[\frac{d}{d\lambda'} Z_{\left(\frac{1}{\lambda'}\right)}\right]_{\lambda'=0} = \left[\frac{6(6 - 3\lambda'^2)}{(3\lambda'^2 + 2\lambda' + 6)^2}\right]_{\lambda'=0} = \frac{1}{6}$$

Let us now consider the admittance function $Y_{(\lambda)}$ of the same network:

$$Y_{(\lambda)} = \frac{6\lambda^2 + 2\lambda + 3}{6\lambda}$$

having one pole at ∞ and one at the origin.

Note, to form: $\lim_{\lambda \to \infty} [(\lambda - \infty)\, Y_{(\lambda)}]$ is absurd, and gives (if something) a residue $= \infty$

Also in this case, however, substitute $\lambda = \frac{1}{\lambda'}$, and investigate at the origin.

Then:

$$Y_{\left(\frac{1}{\lambda'}\right)} = \frac{3\lambda'^2 + 2\lambda' + 6}{6\lambda'}$$

and

$$\lim_{\lambda' \to 0} \left[\lambda' Y_{\left(\frac{1}{\lambda'}\right)}\right] = \lim_{\lambda' \to 0} \left[\frac{3\lambda'^2 + 2\lambda' + 6}{6}\right] = 1$$

Thus, when removing the pole at ∞ from $Y_{(\lambda)}$ we get:

$$Y_{(\lambda)} = \lambda + g_{(\lambda)}$$

where $g_{(\lambda)}$ is $= \frac{2\lambda + 3}{6\lambda}$

Or, we can simply divide the numerator (containing λ in decreasing powers) of $Y_{(\lambda)}$ by its denominator giving:

$$Y_{(\lambda)} = \lambda + \frac{2\lambda + 3}{6\lambda}$$

Example 7. Consider the α-functions:

$$\left.\begin{aligned} \alpha_{11} &= \frac{\lambda}{2\lambda + 1} \\ \alpha_{nn} &= \frac{\lambda^2 + 5\lambda}{2\lambda + 1} \\ \alpha_{1n} &= \frac{\lambda}{2\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots(1)$$

All zeros and poles have proper location, and note α_{nn} only has a pole on the boundary (at ∞), the residue of which is $= +\frac{1}{2}$. Thus the conditions of items (i) (ii) and (iii) of Theorem V[22] are all satisfied.

Further:

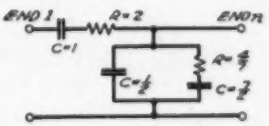
$$\left.\begin{aligned} \alpha_{11(j\omega)} &= \frac{2\omega^2}{4\omega^2+1} + j\omega\,\frac{1}{4\omega^2+1} \\ \alpha_{nn(j\omega)} &= \frac{9\omega^2}{4\omega^2+1} + j\omega\,\frac{2\omega^2+5}{4\omega^2+1} \\ \alpha_{1n(j\omega)} &= \frac{2\omega^2}{4\omega^2+1} + j\omega\,\frac{1}{4\omega^2+1} \end{aligned}\right\} \quad \ldots\ldots(2)$$

Thus, the conditions of items (iv) and (v) of Theorem V are also satisfied.

Then:

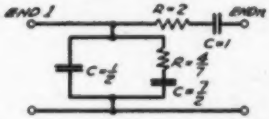
$$\left.\begin{aligned} Y_1 &= \alpha_{11} \qquad\qquad = \frac{\lambda}{2\lambda+1} \\ Y_2 &= \alpha_{nn} - \alpha_{11} = \frac{\lambda^2+4\lambda}{2\lambda+1} = \frac{\lambda}{2} + \frac{3.5}{2\lambda+1} \end{aligned}\right\} \ldots\ldots(3)$$

Consequently, the network representation of the prescribed "positive real" α-matrix is simply the L-structure:

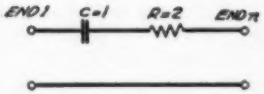


[22] Page 88.

Example 8. Let α_{11} and α_{nn} in Example 7 replace each other, in which case we get the structure:



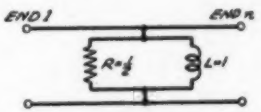
Example 9. Let all α:s be equal to α_{11} in Example 7, and we get the structure:



Example 10. Consider the β:s:

$$\beta_{11} = \beta_{nn} = \beta_{1n} = \frac{\lambda}{2\lambda + 1} \qquad \ldots\ldots (1)$$

From (2) Example 7, we see that the above function is realizable, and we get the structure:



Example 11. Suppose the prescribed functions are:

$$\left.\begin{aligned} \alpha_{11} &= \frac{10\lambda + 10}{100\lambda^2 + 100\lambda + 1} \\ \alpha_{nn} &= \frac{1000\lambda^2 + 10\lambda}{100\lambda^2 + 100\lambda + 1} \\ \alpha_{1n} &= \frac{10\lambda}{100\lambda^2 + 100\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots (1)$$

All three functions are regular in the right half of the λ-plane, i.e. we can limit ourselves to the boundary only. The reality clause is fulfilled and, as no poles lie on the boundary, the only items of Theorem V to consider are (iv) and (v).

These are also satisfied, however, as seen from below:

$$\left.\begin{aligned} \alpha_{11(j\omega)} &= \frac{10}{(1-100\omega^2)^2+(100\omega)^2} + j\omega\,\frac{-990-1000\omega^2}{(1-100\omega^2)^2+(100\omega)^3} \\ \alpha_{nn(j\omega)} &= \frac{10^5\omega^4}{(--)^2+(--)^2} + j\omega\,\frac{10+99000\omega^2}{(--)^2+(--)^2} \\ \alpha_{1n(j\omega)} &= \frac{1000\omega^2}{(--)^2+(--)^2} + j\omega\,\frac{10-1000\omega^2}{(--)^2+(--)^2} \end{aligned}\right\} \ldots(2)$$

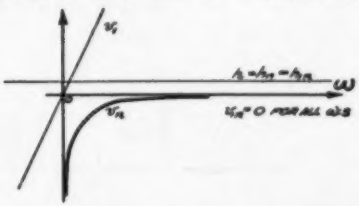
$$r_1 \geq 0;\ r_n \geq 0;\ \text{and}\ (r_1 r_n - r_{1n}^2)^{221a} = 0$$

The corresponding $(r-\omega)$-curves as well as $(x-\omega)$-curves are shown on the next page.

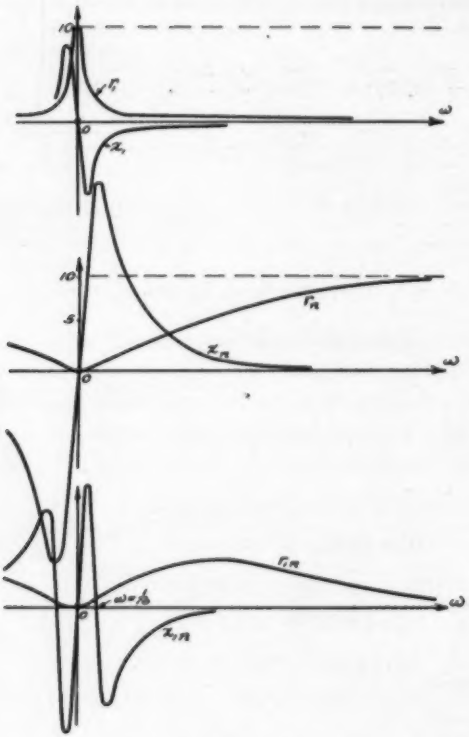
Because no clearness as to the realization possibility of the functions of (1) can be had from these curves, let us transform the α: s into corresponding β: s by the aid of (16).[222] Thus we get:

$$\left.\begin{aligned} \beta_{11} &= \frac{(1000\lambda^2+10\lambda)(100\lambda^2+100\lambda+1)}{100\lambda\,(100\lambda^2+100\lambda+1)} = 10\lambda + \frac{1}{10} \\ \beta_{nn} &= \frac{(10\lambda+10)(100\lambda^2+100\lambda+1)}{100\lambda\,(100\lambda^2+100\lambda+1)} = \frac{1}{10\lambda} + \frac{1}{10} \\ \beta_{1n} &= \frac{10\lambda\,(100\lambda^2+100\lambda+1)}{100\lambda\,(100\lambda^2+100\lambda+1)} = \frac{1}{10} \end{aligned}\right\} \ldots(3)$$

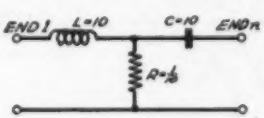
The corresponding $(h-\omega)$-and $(v-\omega)$-curves are:



[221a] Later in Part XI it will be shown that when the determinant of the real parts is = zero, then the corresponding α- or β- matrix can be realized in a simple manner.
[222] Page 31.

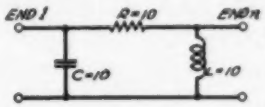


and we see immediately that the β:s of (3) have a network representation in the simple T-structure:

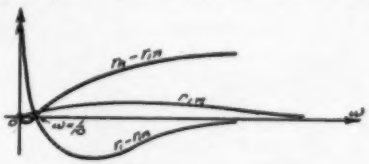


Example 12. Let simply the α:s in (1) of Example 11 be β:s, and let us go through the same procedure of transformations

Then the $(r - \omega)$- and $(x - \omega)$-curves become $(h - \omega)$- and $(v - \omega)$-curves, and vice versa, and we arrive at the π-structure:

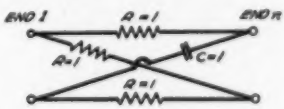


If in Example 11 we had plotted the $(r_1 - r_{1n}) - \omega$, the $(r_n - r_{1n}) - \omega$ and the $(r_{1n} - \omega)$- curves, we would have gotten:



Thus we see that for only one single[223] frequency $\left(\omega = \frac{1}{10}\right)$ the T-structure in Example 11 has an equivalent π-structure, and for the same frequency only, the π-structure in this Example has an equivalent T-structure.

Example 13. Suppose we have the structure:



The α:s are readily found to be:

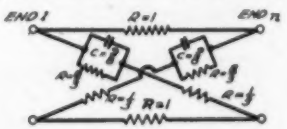
$$\left.\begin{aligned} \alpha_{11} &= \frac{2\lambda + 2}{\lambda + 3} \\ \alpha_{nn} &= \frac{2\lambda + 2}{\lambda + 3} \\ \alpha_{1n} &= \frac{-\lambda + 1}{\lambda + 3} \end{aligned}\right\} \qquad \ldots\ldots(1)$$

[223] Note: actually, as seen from the curves, one more frequency could have been considered, i.e. $\omega = 0$; but at this ω the network representation is simply a resistance bridged across the terminals at end 1.

The driving-point functions are equal, and consequently from (106),[224] with reversed sign for α_{1n} which simply means that the admittances in the horizontal arms and in the cross arms of Fig. 31 are interchanged, we get:

$$\left.\begin{aligned} Y_1 &= \alpha_{11} + \alpha_{1n} = 1 \\ Y_2 &= \alpha_{11} - \alpha_{1n} = \frac{3\lambda + 1}{\lambda + 3} \end{aligned}\right\} \quad \ldots\ldots(2)$$

i.e. we have the equivalent symmetrical lattice structure:



Example 14. Consider the α:s, fulfilling all requirements of Theorem V:

$$\left.\begin{aligned} \alpha_{11} &= \frac{\lambda}{2\lambda + 1} \\ \alpha_{nn} &= \frac{\lambda^2 + 5\lambda}{2\lambda + 1} \\ \alpha_{1n} &= \frac{2\lambda}{2\lambda + 1} \end{aligned}\right\} \quad \ldots\ldots(1)$$

Thus:

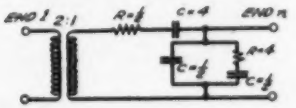
$$\left.\begin{aligned} A &= \frac{\alpha_{nn}}{\alpha_{1n}} &&= \frac{\lambda + 5}{2} \\ B &= \frac{1}{\alpha_{1n}} &&= \frac{2\lambda + 1}{2\lambda} \\ C &= \frac{\alpha_{11}\alpha_{nn} - \alpha_{1n}^2}{\alpha_{1n}} &&= \frac{\lambda^2 + \lambda}{4\lambda + 2} \\ D &= \frac{\alpha_{11}}{\alpha_{1n}} &&= \frac{1}{2} \end{aligned}\right\} \quad \ldots\ldots(2)$$

[224] Page 102.

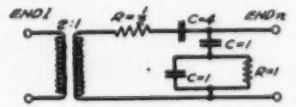
Alternative I:

$$\left.\begin{aligned} Y_1 &= \frac{1}{BD} = \frac{4\lambda}{2\lambda+1} \\ Y_2 &= \frac{C}{D} = \frac{\lambda^2+\lambda}{2\lambda+1} \end{aligned}\right\} \quad \ldots\ldots (3)$$

Further $\frac{1}{D} = 2$, and consequently we get the structure:



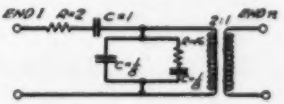
Naturally, several other equivalent structures of the same type could have been obtained. So, for instance, Y_2 could have been realized in a different way, resulting in the structure:



Alternative II:

$$\left.\begin{aligned} Y_1 &= \frac{D}{B} = \frac{\lambda}{2\lambda+1} \\ Y_2 &= CD = \frac{\lambda^2+\lambda}{8\lambda+4} \end{aligned}\right\} \quad \ldots\ldots (4)$$

and we get, for instance, the equivalent structure:



Example 15. Letting the driving-point functions in Example 14 replace each other, we have:

$$\left.\begin{aligned} \alpha_{11} &= \frac{\lambda^2 + 5\lambda}{2\lambda + 1} \\ \alpha_{nn} &= \frac{\lambda}{2\lambda + 1} \\ \alpha_{1n} &= \frac{2\lambda}{2\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots(1)$$

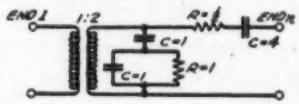
Thus:

$$\left.\begin{aligned} A &= \frac{1}{2} \\ B &= \frac{2\lambda + 1}{2\lambda} \\ C &= \frac{\lambda^2 + \lambda}{4\lambda + 2} \\ D &= \frac{\lambda + 5}{2} \end{aligned}\right\} \qquad \ldots\ldots(2)$$

Alternative I:

$$\left.\begin{aligned} Y_1 = AC &= \frac{\lambda^2 + \lambda}{8\lambda + 4} \\ Y_2 = \frac{A}{B} &= \frac{\lambda}{2\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots(3)$$

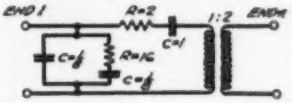
and we have, for instance, the structure:



Alternative II:

$$\left.\begin{aligned} Y_1 &= \frac{C}{A} = \frac{\lambda^2+\lambda}{2\lambda+1} \\ Y_2 &= \frac{1}{AB} = \frac{4\lambda}{2\lambda+1} \end{aligned}\right\} \quad \ldots\ldots(4)$$

giving, for instance, the equivalent structure:



Example 16. Consider the α:s:

$$\left.\begin{aligned} \alpha_{11} &= \frac{\lambda^2+\lambda}{2\lambda+1} \\ \alpha_{nn} &= \frac{9\lambda^2+9\lambda}{2\lambda+1} \\ \alpha_{1n} &= \frac{\lambda}{2\lambda+1} \end{aligned}\right\} \quad \ldots\ldots(1)$$

These functions, when removing the poles at ∞, can be rewritten as:

$$\left.\begin{aligned} \alpha_{11} &= .5\lambda + \frac{.5\lambda}{2\lambda+1} \\ \alpha_{nn} &= 4.5\lambda + \frac{4.5\lambda}{2\lambda+1} \\ \alpha_{1n} &= \frac{\lambda}{2\lambda+1} \end{aligned}\right\} \quad \ldots\ldots(2)$$

All α:s are regular in the right half of the λ-plane, and thus we may solve for their real parts along the boundary only:

$$\left.\begin{aligned} \alpha_{11(j\omega)} &= \frac{\omega^2}{4\omega^2+1} + j\frac{(--)}{(--)} \\ \alpha_{nn(j\omega)} &= \frac{9\omega^2}{4\omega^2+1} + j\frac{(--)}{(--)} \\ \alpha_{1n(j\omega)} &= \frac{2\omega^2}{4\omega^2+1} + j\frac{(--)}{(--)} \end{aligned}\right\} \quad \ldots\ldots(3)$$

We see now that the conditions of all items of Theorem V are satisfied.

The General Circuit Parameters are, by (22):

$$\left.\begin{aligned} A &= 9\lambda + 9 \\ B &= \frac{2\lambda+1}{\lambda} \\ C &= \frac{9\lambda^3 + 18\lambda^2 + 8\lambda}{2\lambda+1} \\ D &= \lambda + 1 \end{aligned}\right\} \quad \ldots\ldots(4)$$

Thus the transformer ratio $\sqrt{\frac{A}{D}}$:1 is 3:1

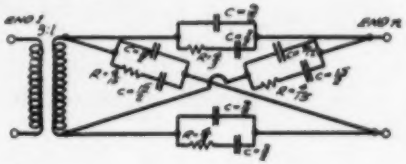
Alternative I:

$$\left.\begin{aligned} Y_1 &= \frac{1}{B}\left(A - \sqrt{\frac{A}{D}}\right) = \frac{9\lambda^2+6\lambda}{2\lambda+1} \\ Y_2 &= \frac{1}{B}\left(A + \sqrt{\frac{A}{D}}\right) = \frac{9\lambda^2+12\lambda}{2\lambda+1} \end{aligned}\right\} \quad \ldots\ldots(5)$$

The above Y:s are "positive real" functions and can be rewritten as:

$$\left.\begin{aligned} Y_1 &= 4.5\lambda + \frac{1.5\lambda}{2\lambda+1} \\ Y_2 &= 4.5\lambda + \frac{7.5\lambda}{2\lambda+1} \end{aligned}\right\} \quad \ldots\ldots(6)$$

and we get the structure:



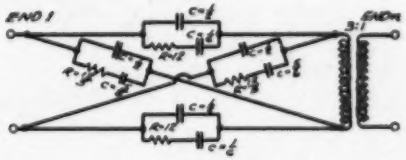
Alternative II:

$$\left.\begin{aligned} Y_1 &= \frac{1}{B}\left(D - \sqrt{\frac{D}{A}}\right) = \frac{3\lambda^2 + 2\lambda}{6\lambda + 3} \\ Y_2 &= \frac{1}{B}\left(D + \sqrt{\frac{D}{A}}\right) = \frac{3\lambda^2 + 4\lambda}{6\lambda + 3} \end{aligned}\right\} \quad \ldots\ldots(7)$$

Also these Y:s are evidently "positive real" functions, and can be rewritten as:

$$\left.\begin{aligned} Y_1 &= \frac{\lambda}{2} + \frac{.5\lambda}{6\lambda + 3} \\ Y_2 &= \frac{\lambda}{2} + \frac{2.5\lambda}{6\lambda + 3} \end{aligned}\right\} \quad \ldots\ldots(8)$$

Consequently we get the structure:



Example 17. Consider the elements of a "positive real" α-matrix:

$$\left.\begin{aligned} \alpha_{11} &= 2 \\ \alpha_{nn} &= 8 \\ \alpha_{1n} &= 3 \end{aligned}\right\} \quad \ldots\ldots(1)^{228}$$

[228] This is a case where it immediately is evident that a transformer has to be included in every one of the infinite number of network representations, because α_{1n} is > one of the driving point "functions" (here being = constants), and all currents are in phase. Naturally, we could as well in our Example have selected λ-functions, but those above, specifying a nonreactive network, are more simple to handle.

The General Circuit "Parameters" are:

$$A = \frac{8}{3},\ B = \frac{1}{3},\ C = \frac{7}{3},\ D = \frac{2}{3}$$

(a) Thus the limiting structure of item (e), Alternative II,[228] is obtained from:

$$\left\|\begin{matrix} \frac{8}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{2}{3} \end{matrix}\right\| = \left\|\begin{matrix} A_1 & B_1 \\ C_1 & D_1 \end{matrix}\right\| \times \left\|\begin{matrix} \frac{3}{2} & 0 \\ 0 & \frac{2}{3} \end{matrix}\right\| \quad \ldots\ldots(2)^{228a}$$

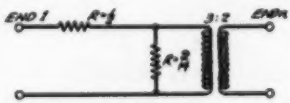
The matrix $\left\|\begin{matrix} A_1 & B_1 \\ C_1 & D_1 \end{matrix}\right\|$, corresponding to the L-structure, is nothing but Y in (121), and in the same equation b^{-1} is nothing but:

$$\left\|\begin{matrix} \frac{2}{3} & 0 \\ 0 & \frac{3}{2} \end{matrix}\right\|$$

Thus:

$$\left\|\begin{matrix} A_1 & B_1 \\ C_1 & D_1 \end{matrix}\right\| = \left\|\begin{matrix} \frac{8}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{2}{3} \end{matrix}\right\| \times \left\|\begin{matrix} \frac{2}{3} & 0 \\ 0 & \frac{3}{2} \end{matrix}\right\| = \left\|\begin{matrix} \frac{16}{9} & \frac{1}{2} \\ \frac{14}{9} & 1 \end{matrix}\right\| \quad \ldots\ldots(3)$$

and we have, by the aid of (22), the structure: $\left(\text{note } n = \frac{3}{2}\right)$



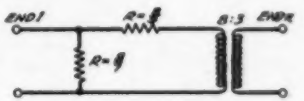
[228] Let us for simplicity consider Alt. II only.

[228a] All the way in this example we let the matrix whose elements we want to determine be: $\left\|\begin{matrix} A_1 & B_1 \\ C_1 & D_1 \end{matrix}\right\|$

(b) The other limiting case, i.e. in item (*f*) Alternative II, is evidently obtained from:

$$\left\| \begin{matrix} A_1 & B_1 \\ C_1 & D_1 \end{matrix} \right\| = \left\| \begin{matrix} \frac{8}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{2}{3} \end{matrix} \right\| \times \left\| \begin{matrix} \frac{3}{8} & 0 \\ 0 & \frac{8}{3} \end{matrix} \right\| = \left\| \begin{matrix} 1 & \frac{8}{9} \\ \frac{7}{8} & \frac{16}{9} \end{matrix} \right\| \quad \ldots\ldots(4)$$

giving the equivalent structure: $\left(\text{note } n = \frac{8}{3}\right)$



Note also, all n:s *between* $\frac{3}{2}$ and $\frac{8}{3}\left(\text{i.e. between } \frac{9}{6} \text{ and } \frac{16}{6}\right)$ give an infinite set of passive structures, because:

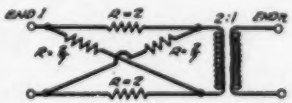
(c) Let for instance, n be 2.
Then:

$$b = \left\| \begin{matrix} 2 & 0 \\ 0 & \frac{1}{2} \end{matrix} \right\| ; \text{ i.e. } b^{-1} = \left\| \begin{matrix} \frac{1}{2} & 0 \\ 0 & 2 \end{matrix} \right\|$$

and we have:

$$\left\| \begin{matrix} A_1 & B_1 \\ C_1 & D_1 \end{matrix} \right\| = \left\| \begin{matrix} \frac{8}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{2}{3} \end{matrix} \right\| \times \left\| \begin{matrix} \frac{1}{2} & 0 \\ 0 & 2 \end{matrix} \right\| = \left\| \begin{matrix} \frac{4}{3} & \frac{2}{3} \\ \frac{7}{6} & \frac{4}{3} \end{matrix} \right\| \quad \ldots\ldots(5)$$

giving the equivalent structure:



(d) Let n be $\frac{15}{6}\left(\text{i.e. } = \frac{5}{2}\right)$.

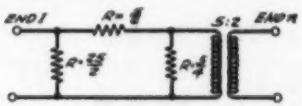
Then:

$$b^{-1} = \begin{Vmatrix} \frac{2}{5} & 0 \\ 0 & \frac{5}{2} \end{Vmatrix}$$

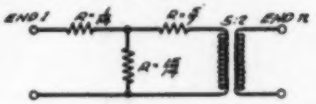
and we have:

$$\begin{Vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{Vmatrix} = \begin{Vmatrix} \frac{8}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{2}{3} \end{Vmatrix} \times \begin{Vmatrix} \frac{2}{5} & 0 \\ 0 & \frac{5}{2} \end{Vmatrix} = \begin{Vmatrix} \frac{16}{15} & \frac{5}{6} \\ \frac{14}{15} & \frac{10}{6} \end{Vmatrix} \quad \ldots\ldots(6)$$

giving the equivalent structure:

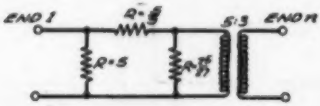


(d′) Under (d) we could as well by (22) have solved for the β:s giving the equivalent structure:



(e) Let n be $\frac{10}{6}$.

Then, solving for the α:s we in an analogous manner get the structure:



When letting n be $> \frac{16}{6}$ or $< \frac{9}{6}$, however, then the Y-matrix cannot be realized in a single step. Two transformers will then

be contained in our structure, and when they are merged together into a single one we are back to one of those networks which are directly obtained for $\frac{16}{6} \geq n \geq \frac{9}{6}$, *i.e. these two n-values form the limits.*

Example 18. Consider the elements of a "positive real" matrix:

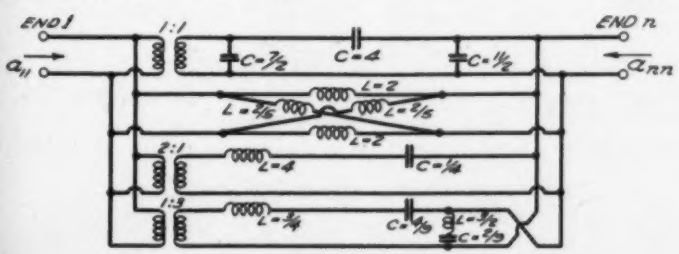
$$\left.\begin{aligned} f_{11(\lambda)} &= \frac{15\lambda^6 + 89\lambda^4 + 87\lambda^2 + 9}{2\lambda^5 + 8\lambda^3 + 6\lambda} \\ f_{nn(\lambda)} &= \frac{19\lambda^6 + 91\lambda^4 + 97\lambda^2 + 9}{2\lambda^5 + 8\lambda^3 + 6\lambda} \\ f_{1n(\lambda)} &= \frac{8\lambda^6 + 30\lambda^4 + 36\lambda^2 + 6}{2\lambda^5 + 8\lambda^3 + 6\lambda} \end{aligned}\right\} \quad \ldots\ldots(1)$$

The above functions fulfil all requirements of item (a) Part VIII, and after expansion according to (146) they become:

$$\left.\begin{aligned} f_{11(\lambda)} &= \frac{15}{2}\lambda + \frac{3}{2\lambda} + \frac{\lambda}{\lambda^2+1} + \frac{12\lambda}{\lambda^2+3} \\ f_{nn(\lambda)} &= \frac{19}{2}\lambda + \frac{3}{2\lambda} + \frac{4\lambda}{\lambda^2+1} + \frac{2\lambda}{\lambda^2+3} \\ f_{1n(\lambda)} &= 4\lambda + \frac{1}{\lambda} + \frac{2\lambda}{\lambda^2+1} - \frac{4\lambda}{\lambda^2+3} \end{aligned}\right\} \quad \ldots\ldots(2)$$

(1) Suppose the f:s are the three short-circuit admittances (the α:s).

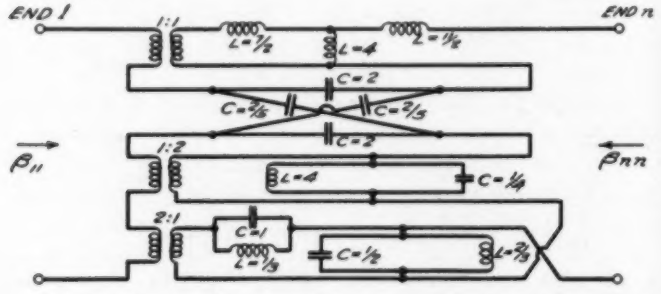
Then, by the aid of the foregoing, the total network representation is, for instance, the one shown below:



Note, the necessity of the transformers as indicated. Those having ratios different from 1:1 must under all circumstances stay in. Further, should here the 1:1 ratio transformer, contained in the upper component structure, be omitted, then the inductance coil of the lower horizontal arm of the symmetrical lattice-structure would be shorted out and the structure, as a whole, would not be a network representation of the total prescribed α-functions.

(2) Now, suppose the f:s are the three open-circuit impedances (the β:s).

Then, the network-representation is, for instance, the one shown below:



Again, note here the necessity of the transformers. Should in this case the 1:1 ratio transformer be omitted, then the condenser of the upper horizontal arm of the symmetrical lattice-structure would be shorted out, and again the total structure would not be a true network representation of the total functions.

Example 19. Consider the elements of a "positive real" matrix:

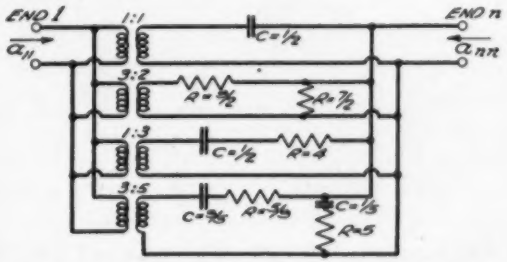
$$\left.\begin{aligned} f_{11(\lambda)} &= \frac{\lambda^3 + 20\lambda^2 + 16\lambda + 2}{2\lambda^2 + 3\lambda + 1} \\ f_{nn(\lambda)} &= \frac{\lambda^3 + 22\lambda^2 + 27\lambda + 8}{2\lambda^2 + 3\lambda + 1} \\ f_{1n(\lambda)} &= \frac{\lambda^3 + 15\lambda^2 + 14\lambda + 3}{2\lambda^2 + 3\lambda + 1} \end{aligned}\right\} \quad \ldots\ldots(1)$$

The above functions fulfil all requirements of item (b) Part VIII, and after being expanded, according to (149), they become:

$$\left.\begin{aligned} f_{11(\lambda)} &= \frac{\lambda}{2} + 2 + \frac{4.5\lambda}{2\lambda+1} + \frac{5\lambda}{\lambda+1} \\ f_{nn(\lambda)} &= \frac{\lambda}{2} + 8 + \frac{.5\lambda}{2\lambda+1} + \frac{2\lambda}{\lambda+1} \\ f_{1n(\lambda)} &= \frac{\lambda}{2} + 3 + \frac{1.5\lambda}{2\lambda+1} + \frac{3\lambda}{\lambda+1} \end{aligned}\right\} \quad \ldots\ldots(2)$$

(1) Suppose the f:s are the three short-circuit admittances (the α:s).

Then, by the aid of the foregoing their total network representation is, for instance, the one shown below, although naturally each one of the different component-structures can be replaced with some other equivalent structure.



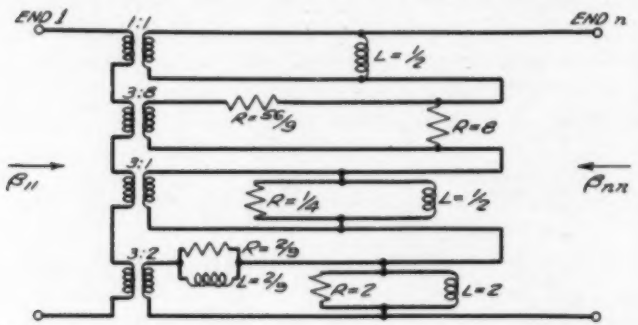
The upper structure, for instance, could as well have been selected as a symmetrical one having equal admittances in each one of two horizontal branches.

If symmetrical or not its 1:1 ratio transformer may in this case be omitted.

(2) Now, suppose the f:s are the three open-circuit impedances (the β:s).

Then, the network representation is, for instance, the one shown below, although again some of the different component structures

can be replaced with some other equivalent structures. Also in this case the 1:1 ratio transformer may be omitted.



Example 20. Consider the elements of a "positive real" matrix:

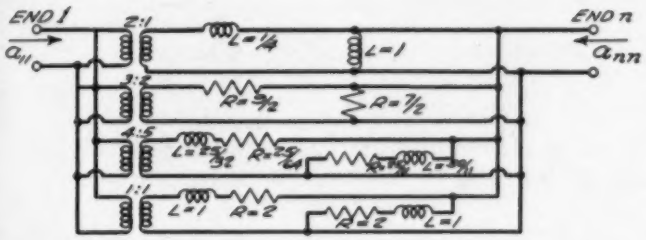
$$\left.\begin{aligned} f_{11(\lambda)} &= \frac{4\lambda^3 + 18\lambda^2 + 18\lambda + 2}{2\lambda^3 + 5\lambda^2 + 2\lambda} \\ f_{nn(\lambda)} &= \frac{16\lambda^3 + 57\lambda^2 + 49\lambda + 10}{2\lambda^3 + 5\lambda^2 + 2\lambda} \\ f_{1n(\lambda)} &= \frac{6\lambda^3 + 24.2\lambda^2 + 23.4\lambda + 4}{2\lambda^3 + 5\lambda^2 + 2\lambda} \end{aligned}\right\} \quad \ldots\ldots(1)$$

The above functions fulfil all requirements of item (c) Part VIII, and after being expanded, according to (151), they become:

$$\left.\begin{aligned} f_{11(\lambda)} &= \frac{1}{\lambda} + 2 + \frac{4}{2\lambda + 1} + \frac{1}{\lambda + 2} \\ f_{nn(\lambda)} &= \frac{5}{\lambda} + 8 + \frac{3}{2\lambda + 1} + \frac{2}{\lambda + 2} \\ f_{1n(\lambda)} &= \frac{2}{\lambda} + 3 + \frac{3.2}{2\lambda + 1} + \frac{1}{\lambda + 2} \end{aligned}\right\} \quad \ldots\ldots(2)$$

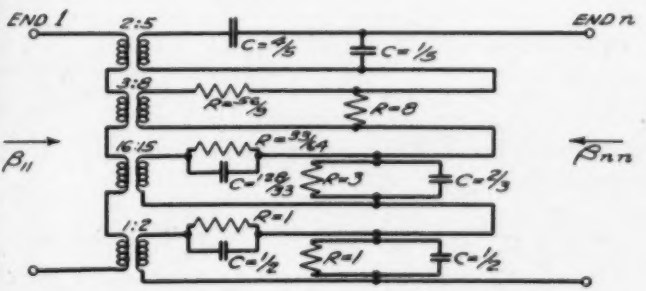
(1) Suppose the f:s are the three short-circuit admittances (the α:s).

Then, their total network representation is, for instance, the one shown below, although again naturally other component structures could have been selected. The 1:1 ratio transformer, contained in the lower component-structure, may in this case be omitted.



(2) Now, suppose the f:s are the three open-circuit impedances (the β:s).

Then, the network representation is, for instance, the one shown below. Again different component structures could have been selected; but note that, if using those as indicated, all transformers must stay in, because each one has a ratio different from 1:1. Had the other L-structure been selected for the lower component structure, then the ratio of its transformer would have become 1:1 and this transformer could then have been omitted.



Example 21. Let us start out from the ω-function (supposed to be the real part along the boundary of a rational, fractional function of a complex variable):

$$F_{(\omega)} = \frac{G_{(\omega)}}{H_{(\omega)}} = \frac{2}{\omega^6 + \omega^4 + 4\omega^2 + 4} \qquad \ldots\ldots(1)$$

and find corresponding λ-function: $f_{(\lambda)} = \frac{g_{(\lambda)}}{h_{(\lambda)}}$!

The denominator of (1) can be factored as:

$$H_{(\omega)} = (\omega^2 + 1)(\omega^4 + 4) \qquad \ldots\ldots(2)$$

i.e. one of the roots[227] is real and negative, and the other two roots are imaginary (= complex with the real part zero).

Thus σ_r is $= -1$, and by (158):

$$\lambda_r = -\sqrt{-(-1)} = -1$$

i.e. the factor $(\omega^2 + 1)$ of $H_{(\omega)}$ corresponds to the factor $(\lambda + 1)$ of "$h_{(\lambda)}$."

Further, the factor $(\omega^4 + 4)$ of $H_{(\omega)}$ generates, by the aid of (159) and (160), the λ-expression:

$$+\sqrt{1}\lambda^2 + \left[+\sqrt{0+2\sqrt{1\times 4}}\right]\lambda + \sqrt{4} = \lambda^2 + 2\lambda + 2$$

Then:

$$(\lambda + 1)\ (\lambda^2 + 2\lambda + 2) = \lambda^3 + 3\lambda^2 + 4\lambda + 2$$

As to the proportionality factor, we get from (161):

$$B_3 = 1 = b^2, \text{ i.e. } |b_3| = 1$$

which means that said factor is in this case equal to $+1$, and consequently:

$$h_{(\lambda)} = b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = \lambda^3 + 3\lambda^2 + 4\lambda + 2 \ldots(3)$$

[227] Note, in the whole Example roots are in terms of ω^2, and not of ω.

Then by the aid of $G_{(\omega^2)}$ and (162):

$$\left.\begin{aligned} A_0 &= 2 = a_0 b_0 \\ A_1 &= 0 = a_1 b_1 - (a_0 b_2 + a_2 b_0) \\ A_2 &= 0 = a_2 b_2 - (a_1 b_3 + a_3 b_1) \\ A_3 &= 0 = a_3 b_3 \end{aligned}\right\} \quad \ldots\ldots(4)$$

from which we get:

$$a_0 = 1,\ a_1 = .9,\ a_2 = .3 \text{ and } a_3 = 0$$

i.e.

$$g_{(\lambda)} = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = .3\lambda^2 + .9\lambda + 1$$

and finally:

$$f_{(\lambda)} = \frac{.3\lambda^2 + .9\lambda + 1}{\lambda^3 + 3\lambda^2 + 4\lambda + 2} \quad \ldots\ldots(5)$$

which is readily seen to be a "positive real" function.

As a check by solving for $Re\, f_{(j\omega)}$ we are back to (1).

Example 22. Let us consider the "positive real" α-matrix (say) having the rather comprehensive elements:

$$\left.\begin{aligned} \alpha_{11} &= \frac{\lambda^5 + 7\lambda^4 + 22\lambda^3 + 38\lambda^2 + 37\lambda + 15}{\lambda^4 + 3\lambda^3 + 4\lambda^2 + 2\lambda} \\ \alpha_{nn} &= \frac{4\lambda^4 + 20\lambda^3 + 34\lambda^2 + 24\lambda + 8}{\lambda^4 + 3\lambda^3 + 4\lambda^2 + 2\lambda} \\ \alpha_{1n} &= \frac{4\lambda^4 + 4\lambda^3 - 2\lambda^2 - 4\lambda - 2}{\lambda^4 + 3\lambda^3 + 4\lambda^2 + 2\lambda} \end{aligned}\right\} \quad \ldots\ldots(1)$$

The α:s of (1) can be factored as:

$$\left.\begin{aligned} \alpha_{11} &= \frac{(\lambda+1)\,[(\lambda+1)^2 + 2]\,[(\lambda+2)^2 + 1]}{\lambda(\lambda+1)\,[(\lambda+1)^2 + 1]} \\ \alpha_{nn} &= \frac{(\lambda+2)^2\,[(2\lambda+1)^2 + 1]}{\lambda(\lambda+1)\,[(\lambda+1)^2 + 1]} \\ \alpha_{1n} &= \frac{(\lambda+1)\,(\lambda-1)\,[(2\lambda+1)^2 + 1]}{\lambda(\lambda+1)\,[(\lambda+1)^2 + 1]} \end{aligned}\right\} \quad \ldots\ldots(2)$$

from which it is clear that the location of zeros and poles is correct, and consequently we can limit our efforts to the boundary.

All α:s have common poles at the origin, at $\lambda = -1$ on the negative real axis, and at two conjugate points in the interior of the left half of the λ-plane. In addition, α_{11} alone has a pole at ∞.

1) The reality clause and the location requirement for the poles as stated in item (i), Theorem V, are fulfilled.

2) As to the residue conditions, we have:

For the pole at ∞:

$$k_1 = \lim_{\lambda' \to 0} \left[\lambda' \alpha_{11}\left(\tfrac{1}{\lambda'}\right) \right] \text{is} = 1, \text{ i.e. } > \text{zero} \quad \ldots\ldots (3)$$

Corresponding "residues" of the other two α:s, which have no pole at ∞, are zeros, i.e: the determinant condition:

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \geq \text{zero is, with the equal sign, satisfied.}$$

For the common pole at the origin:

$$\left.\begin{aligned} k_1' &= \lim_{\lambda \to 0} (\lambda \alpha_{11}) \text{ is } = 7.5 \text{ i.e. } > \text{zero} \\ k_n' &= \lim_{\lambda \to 0} (\lambda \alpha_{nn}) \text{ is } = 4 \text{ , i.e. } > \text{zero} \\ k_{1n}' &= \lim_{\lambda \to 0} (\lambda \alpha_{1n}) \text{ is } = -1 \end{aligned}\right\} \quad \ldots\ldots (4)$$

Thus the determinant $\begin{vmatrix} 7.5 & -1 \\ -1 & 4 \end{vmatrix}$ is $= 29$, i.e. $>$ zero.

Consequently, also for these poles the conditions of items (ii) and (iii) of Theorem V are satisfied.

3) Finally, we come to the question of positiveness of certain real parts and of the determinant of the real parts. We find:

$$\left.\begin{aligned} Re\,\alpha_{11(j\omega)} &= r_1 = \frac{2(2\omega^6+\omega^4+6\omega^2+7)}{(\omega^3-4\omega)^2+(2-3\omega^2)^2} \\ Re\,\alpha_{nn(j\omega)} &= r_n = \frac{2(2\omega^6+5\omega^4+16\omega^2+8)}{(\omega^3-4\omega)^2+(2-3\omega^2)^2} \\ Re\,\alpha_{1n(j\omega)} &= r_{1n} = \frac{2\omega^2(2\omega^4-\omega^2-3)}{(\omega^3-4\omega)^2+(2-3\omega^2)^2} \end{aligned}\right\} \quad \ldots\ldots(5)$$

It can readily be found that r_1 and r_n are each $\geq$ zero[228], for all real ω:s.

The r-determinant becomes:

$$(r_1 r_n - r_{1n}^2) = \frac{8(8\omega^{10}+30\omega^8+35\omega^6+65\omega^4+80\omega^2+28)}{(\omega^6+\omega^4+4\omega^2+4)^2} \quad \ldots(6)$$

and also, this expression is found to be $\geq$ zero[228], for all real ω:s.

Consequently, also the conditions of the last two items of Theorem V are satisfied, and *from now on we know that the* α:s *of* (1) *actually are elements of a "positive real" matrix.*

We now proceed to find a network representation of the α*-functions of* (1).

Let us first remove the poles from the boundary and realize them separately.

From (3) and (4) we already know the coefficients of these removed poles (being nothing but corresponding residues), and (1) goes over into:

$$\left.\begin{aligned} \alpha_{11} &= \lambda + \frac{7.5}{\lambda} + \frac{4\lambda^3+10.5\lambda^2+13.5\lambda+7}{\lambda^3+3\lambda^2+4\lambda+2} = \overset{0}{\alpha}_{11} + \alpha'_{11} + \alpha''_{11} \\ \alpha_{nn} &= \frac{4}{\lambda} + \frac{4\lambda^3+14\lambda^2+22\lambda+10}{\lambda^3+3\lambda^2+4\lambda+2} = \alpha'_{nn} + \alpha''_{nn} \\ \alpha_{1n} &= \frac{-1}{\lambda} + \frac{4\lambda^3+5\lambda^2+\lambda}{\lambda^3+3\lambda^2+4\lambda+2} = \alpha'_{1n} + \alpha''_{1n} \end{aligned}\right\} \ldots(7)$$

[228] As seen, it is rather difficult to see if these expressions are $\geq$ zero, for $0 < \omega^2 < +\infty$ but by the aid of item (g) of Part II (illustrated in Example 3) it is clear.

α^0_{11} simply corresponds to a condenser, of 1 farad capacity, bridged across the terminals at end 1 of the total network representation.

The α':s are readily seen to be realizable into any one of an infinite set of structures, because they fall under item (h) of Part VII. Let us, however, select the combination of a transformer and an L-structure, as shown in Fig. 37.[229]

$$\left.\begin{aligned} \alpha'_{11} &= \frac{7.5}{\lambda} \\ \alpha'_{nn} &= \frac{4}{\lambda} \\ \alpha'_{1n} &= \frac{-1}{\lambda} \end{aligned}\right\} \qquad \ldots\ldots(8)$$

The General Circuit Parameters of (8) are:

$$\left.\begin{aligned} A &= -4 \\ B &= -\lambda \\ C &= -\frac{29}{\lambda} \\ D &= -7.5 \end{aligned}\right\} \qquad \ldots\ldots(9)$$

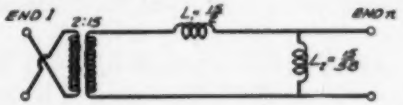
Thus $n:1 = \left|\frac{1}{D}\right| :1 = 2:15$

Further, by (125):[230]

$$\left.\begin{aligned} Y_1 &= \frac{1}{BD} = \frac{2}{15\lambda} \\ Y_2 &= \frac{C}{D} = \frac{58}{15\lambda} \end{aligned}\right\} \qquad \ldots\ldots(10)$$

[229] Page 119.
[230] Page 120.

giving the component structure:



Then, we will have to realize the α'':s of (7).

Their real parts are the same as those of the α:s of (1), i.e. they are those given by equations (5), rewritten below somewhat modified:

$$\left.\begin{aligned} r_1'' &= \frac{4\omega^6 + 2\omega^4 + 12\omega^2 + 14}{(\omega^2+1)(\omega^4+4)} \\ r_n'' &= \frac{4\omega^6 + 4\omega^4 + 30\omega^2 + 20}{(\omega^2+1)(\omega^4+4)} \\ r_{1n}'' &= \frac{4\omega^6 - 2\omega^4 - 6\omega^2}{(\omega^2+1)(\omega^4+4)} \end{aligned}\right\} \quad \ldots\ldots (11)$$

Let us split up the numerators of (11) as shown below:

$$\left.\begin{aligned} r_1'' &= \frac{4\omega^6 - \omega^4 + 9\omega^2 + 14}{(\omega^2+1)(\omega^4+4)} + \frac{3(\omega^4+\omega^2)}{(\omega^2+1)(\omega^4+4)} = r_1''' + r_1'^{\mathrm{v}} \\ r_n'' &= \frac{4\omega^6 - \omega^4 + 9\omega^2 + 14}{(\omega^2+1)(\omega^4+4)} + \frac{5\omega^4 + 21\omega^2 + 6}{(\omega^2+1)(\omega^4+4)} = r_n''' + r_n'^{\mathrm{v}} \\ r_{1n}'' &= \frac{4\omega^6 - 4\omega^4 - 8\omega^2}{(\omega^2+1)(\omega^4+4)} + \frac{2(\omega^4+\omega^2)}{(\omega^2+1)(\omega^4+4)} = r_{1n}''' + r_{1n}'^{\mathrm{v}} \end{aligned}\right\} (12)$$

It can now readily be found that in (12):

$$r_1''' \geq 0;\ r_n''' \geq 0 \text{ and } (r_1''' r_n''' - r_{1n}^{2'''}) \geq 0$$

further:

$$r_1'^{\mathrm{v}} \geq 0;\ r_n'^{\mathrm{v}} \geq 0 \text{ and } (r_1'^{\mathrm{v}} r_n'^{\mathrm{v}} - r_{1n}^{2'\mathrm{v}}) \geq 0$$

The regularity clause is not violated, because the denominators are unchanged, i.e. the denominators of the functions of (12) have their correspondence in the denominators of the α'':s of (7).

Then, the numerators can, by the aid of equations (162),[231] readily be found, and we arrive at the following component families:

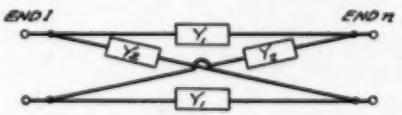
$$\left.\begin{aligned} \alpha_{11}^{'''} &= \frac{4\lambda^3 + 9\lambda^2 + 12\lambda + 7}{\lambda^3 + 3\lambda^2 + 4\lambda + 2} \\ \alpha_{nn}^{'''} &= \frac{4\lambda^3 + 9\lambda^2 + 12\lambda + 7}{\lambda^3 + 3\lambda^2 + 4\lambda + 2} \\ \alpha_{1n}^{'''} &= \frac{4\lambda^3 + 4\lambda^2}{\lambda^3 + 3\lambda^2 + 4\lambda + 2} \end{aligned}\right\} \quad \ldots\ldots(13)$$

and:

$$\left.\begin{aligned} \alpha_{11}^{'v} &= \frac{1.5\lambda^2 + 1.5\lambda}{\lambda^3 + 3\lambda^2 + 4\lambda + 2} \\ \alpha_{nn}^{'v} &= \frac{5\lambda^2 + 10\lambda + 3}{\lambda^3 + 3\lambda^2 + 4\lambda + 2} \\ \alpha_{1n}^{'v} &= \frac{\lambda^2 + \lambda}{\lambda^3 + 3\lambda^2 + 4\lambda + 2} \end{aligned}\right\} \quad \ldots\ldots(14)$$

The α:s of (13) and (14) above have evidently network representations in a symmetrical lattice-structure and in, for instance, the structure shown in Fig. 37[232] respectively.[233]

1) Let us first consider (13).

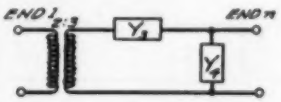


[231] Page 146.
[232] Page 119.
[233] This can be seen from (12) of this Example already.

By the aid of (106)[234] we obtain the following expressions for the Y:s:

$$\left.\begin{aligned} Y_1 &= \alpha_{11}''' - \alpha_{1n}''' = \frac{5\lambda + 7}{\lambda^2 + 2\lambda + 2} \\ Y_2 &= \alpha_{11}''' + \alpha_{1n}''' = \frac{8\lambda^2 + 5\lambda + 7}{\lambda^2 + 2\lambda + 2} \end{aligned}\right\} \quad \ldots\ldots(15)$$

2) Then, let us consider (14).



By the aid of (22) we get the General Circuit Parameters:

$$\left.\begin{aligned} A &= \frac{5\lambda^2 + 10\lambda + 3}{\lambda^2 + \lambda} \\ B &= \frac{\lambda^2 + 2\lambda + 2}{\lambda} \\ C &= \frac{6.5\lambda^2 + 14\lambda + 4.5}{\lambda^3 + 3\lambda^2 + 4\lambda + 2} \\ D &= \frac{3}{2} \end{aligned}\right\} \quad \ldots\ldots(16)$$

Thus:

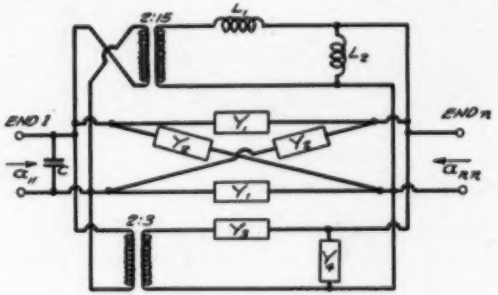
$$n:1 = \frac{1}{D}:1 = 2:3$$

Further, from (125):

$$\left.\begin{aligned} Y_3 &= \frac{1}{BD} = \frac{2\lambda}{3\lambda^2 + 6\lambda + 6} \\ Y_4 &= \frac{C}{D} = \frac{13\lambda^2 + 28\lambda + 9}{3\lambda^3 + 9\lambda^2 + 12\lambda + 6} \end{aligned}\right\} \quad \ldots\ldots(17)$$

[234] Page 102.

Consequently, the total network representation of the α:s of (1) is, for instance:



where:

$$\left.\begin{aligned} C &= 1 \\ L_1 &= \frac{15}{2} \\ L_2 &= \frac{15}{58} \\ Y_1 &= \frac{5\lambda + 7}{\lambda^2 + 2\lambda + 2} \\ Y_2 &= \frac{8\lambda^2 + 5\lambda + 7}{\lambda^2 + 2\lambda + 2} \\ Y_3 &= \frac{2\lambda}{3\lambda^2 + 6\lambda + 6} \\ Y_4 &= \frac{13\lambda^2 + 28\lambda + 9}{3\lambda^3 + 9\lambda^2 + 12\lambda + 6} \end{aligned}\right\} \quad \ldots\ldots (18)$$

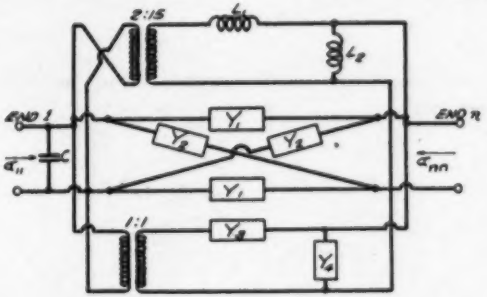
Note: in (18) C, L_1, and L_2 are all positive elements, and all the Y:s are "positive real" functions.

On the other hand, equations (11) could have been split up in numerous other ways and for each way of splitting we would get a different total network representation.

So, for instance, if splitting (11) as:

$$\left.\begin{aligned} r_1'' &= \frac{4\omega^6+\omega^4+11\omega^2+14}{(\omega^2+1)(\omega^4+4)} + \frac{\omega^4+\omega^2}{(\omega^2+1)(\omega^4+4)} = r_1''' + r_1'^{\vee} \\ r_n'' &= \frac{4\omega^6+\omega^4+11\omega^2+14}{(\omega^2+1)(\omega^4+4)} + \frac{3\omega^4+19\omega^2+6}{(\omega^2+1)(\omega^4+4)} = r_n''' + r_n'^{\vee} \\ r_{1n}'' &= \frac{4\omega^6-3\omega^4-7\omega^2}{(\omega^2+1)(\omega^4+4)} + \frac{\omega^4+\omega^2}{(\omega^2+1)(\omega^4+4)} = r_{1n}''' + r_{1n}'^{\vee} \end{aligned}\right\} . \quad (19)$$

and if then going through the same procedure as for the first case, we would arrive at the total equivalent network representation:



where.

$$\left.\begin{aligned} C &= 1 \\ L_1 &= \frac{15}{2} \\ L_2 &= \frac{15}{58} \\ Y_1 &= \frac{8\lambda^3+14.5\lambda^2+13.5\lambda+7}{\lambda^3+3\lambda^2+4\lambda+2} \\ Y_2 &= \frac{5.5\lambda^2+12.5\lambda+7}{\lambda^3+3\lambda^2+4\lambda+2} \\ Y_3 &= \frac{.5\lambda^2+.5\lambda}{\lambda^3+3\lambda^2+4\lambda+2} \\ Y_4 &= \frac{3.5\lambda^2+8.5\lambda+3}{\lambda^3+3\lambda^2+4\lambda+2} \end{aligned}\right\} \quad \ldots\ldots (20)$$

Again, note that C, L_1 and L_2 are all positive elements and all the Y:s are "positive real" functions.

Example 23. Consider the elements of a "positive real" α-matrix:

$$\left.\begin{aligned} \alpha_{11} &= \frac{2\lambda^3 + 6\lambda^2 + 5\lambda + 2}{\lambda^2 + 2\lambda + 2} \\ \alpha_{nn} &= \frac{8\lambda^3 + 18\lambda^2 + 18\lambda + 1}{\lambda^2 + 2\lambda + 2} \\ \alpha_{1n} &= \frac{3\lambda^3 + 7\lambda^2 + 7\lambda - 1}{\lambda^2 + 2\lambda + 2} \end{aligned}\right\} \quad \ldots\ldots(1)$$

After removal of inherent poles on the boundary (in this case at ∞), we get:

$$\left.\begin{aligned} \alpha_{11} &= 2\lambda + \frac{\lambda^2 + \lambda + 2}{\lambda^2 + 2\lambda + 2} \\ \alpha_{nn} &= 8\lambda + \frac{2\lambda^2 + 2\lambda + 1}{\lambda^2 + 2\lambda + 2} \\ \alpha_{1n} &= 3\lambda + \frac{\lambda^2 + \lambda - 1}{\lambda^2 + 2\lambda + 2} \end{aligned}\right\} \quad \ldots\ldots(2)$$

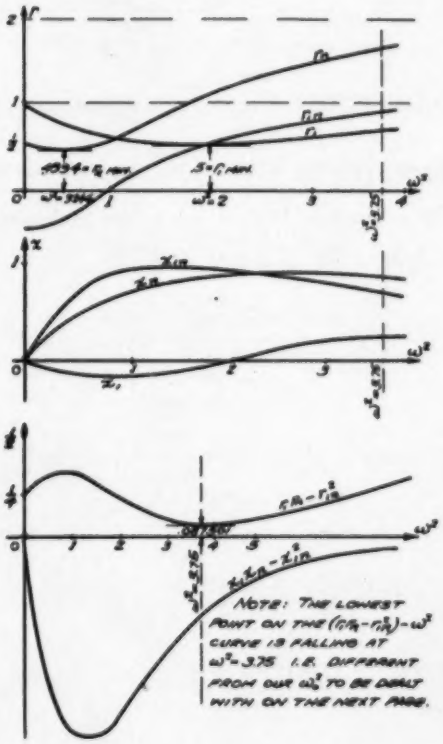
The left component family can now separately be realized, and considering the right component family only we find these α:s, along the boundary, to be:

$$\left.\begin{aligned} \alpha_{11(j\omega)} &= \frac{\omega^4 - 2\omega^2 + 4}{\omega^4 + 4} + j\omega\frac{\omega^2 - 2}{\omega^4 + 4} = r_1 + jx_1 \\ \alpha_{nn(j\omega)} &= \frac{2\omega^4 - \omega^2 + 2}{\omega^4 + 4} + j\omega\frac{2\omega^2 + 2}{\omega^4 + 4} = r_n + jx_n \\ \alpha_{1n(j\omega)} &= \frac{\omega^4 + \omega^2 - 2}{\omega^4 + 4} + j\omega\frac{\omega^2 + 4}{\omega^4 + 4} = r_{1n} + jx_{1n} \end{aligned}\right\} \ldots(3)$$

Thus, the r- and x-determinants, respectively, are:

$$\left.\begin{aligned}(r_1r_n - r_{1n}^2) &= \frac{\omega^8 - 7\omega^6 + 15\omega^4 - 4\omega^2 + 4}{\omega^8 + 8\omega^4 + 16} \\ (x_1x_n - x_{1n}^2) &= \frac{\omega^2(\omega^4 - 10\omega^2 - 20)}{\omega^8 + 8\omega^4 + 16}\end{aligned}\right\} \quad \ldots\ldots (4)$$

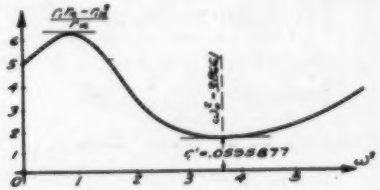
Below the r:s, the x:s and the r- and x-determinants respectively are plotted versus positive ω^2 − values.


r
x
NOTE: THE LOWEST POINT ON THE $(r_1r_n - r_{1n}^2) - \omega^2$ CURVE IS FALLING AT $\omega^2 = 3.75$ I.E. DIFFERENT FROM OUR ω_0^2 TO BE DEALT WITH ON THE NEXT PAGE.

Dividing the r-determinant by r_n we get:

$$\frac{r_1 r_n - r_{1n}^2}{r_n} = \frac{\omega^8 - 7\omega^6 + 15\omega^4 - 4\omega^2 + 4}{2\omega^8 - \omega^6 + 10\omega^4 - 4\omega^2 + 8} \quad \ldots\ldots (5)$$

The corresponding curve is:



The derivative: $\frac{d}{d(\omega^2)}\left[\frac{r_1 r_n - r_{1n}^2}{r_n}\right]$, equated to zero, gives us:

$$13\omega^{12} - 40\omega^{10} - 43\omega^8 + 48\omega^6 - 176\omega^4 + 160\omega^2 - 16 = 0 \ldots (6)$$

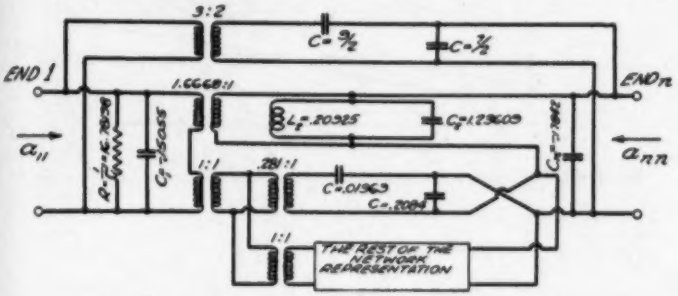
Proceeding as outlined in Part X we then find, when considering the $\frac{r_1 r_n - r_{1n}^2}{r_n} - \omega^2$ curve;

$$\left.\begin{aligned} \omega_0^2 &= 3.8661 \\ r_1' &= .0595877 \\ x_1' &= -.296030 \\ x_n' &= -.350828 \end{aligned}\right\}$$

Thus, the members of the right component family of (2) can be written:

$$\left.\begin{aligned} \alpha_{11} &= .0595877 + \\ &\quad - \frac{.296030}{\sqrt{3.8661}}\lambda + \frac{.15055\lambda^3 + 1.24152\lambda^2 + 1.18193\lambda + 1.88082}{\lambda^2 + 2\lambda + 2} \\ \alpha_{nn} &= -\frac{.350828}{\sqrt{3.8661}}\lambda + \frac{.17842\lambda^3 + 2.35685\lambda^2 + 2.35685\lambda + 1}{\lambda^2 + 2\lambda + 2} \\ \alpha_{1n} &= \frac{\lambda^2 + \lambda - 1}{\lambda^2 + 2\lambda + 2} \end{aligned}\right\} \ldots (7)$$

Continuing with inversions as outlined we then find the realization for the first cycle to be, for instance, the one shown below:



When then starting the second cycle the α:s to consider are:

$$\left.\begin{aligned}\alpha_{11} &= \frac{.81879\lambda^2 + 1.27875\lambda + .97301}{.51723\lambda^2 + 1.03425\lambda + 1.03387}\\ \alpha_{nn} &= \frac{1.18214\lambda^2 + 1.36758\lambda + .51730}{.51723\lambda^2 + 1.03425\lambda + 1.03387}\\ \alpha_{1n} &= \frac{.29521\lambda^2 - .007656\lambda - .51734}{.51723\lambda^2 + 1.03425\lambda + 1.03387}\end{aligned}\right\} \quad \ldots\ldots(8)$$

Thus:

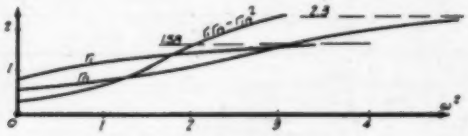
$$\left.\begin{aligned}y_{11} &= \frac{.42350\omega^4 - .02724\omega^2 + 1.00596}{.26752\omega^4 - .00018\omega^2 + 1.06888} + j\omega\,\frac{.18543\omega^2 + .31573}{.26752\omega^4 - .00018\omega^2 + 1.06888}\\ y_{nn} &= \frac{.61143\omega^4 - .07532\omega^2 + .53482}{.26752\omega^4 - \ldots\ldots\ldots\ldots} + j\omega\,\frac{.51527\omega^2 + .87888}{.26752\omega^4 - \ldots\ldots\ldots\ldots}\\ y_{1n} &= \frac{.15269\omega^4 - .04553\omega^2 - .53486}{.26752\omega^4 - \ldots\ldots\ldots\ldots} + j\omega\,\frac{.30927\omega^2 + .52714}{.26752\omega^4 - \ldots\ldots\ldots\ldots}\end{aligned}\right\} \ldots(9)$$

and the r-determinant is found to be:

$$(r_1r_n - r_{1n}^2) =$$

$$= \frac{.23563\omega^6 - .03465\omega^6 + 1.00487\omega^4 - .13902\omega^2 + .25193}{.071566\omega^8 - .000096\omega^6 + .57189\omega^4 - .000384\omega^2 + 1.14250} \ldots(10)$$

The graphical representations of r_1, r_n and $(r_1 r_n - r_{1n}^2)$ versus ω^2 are shown below:

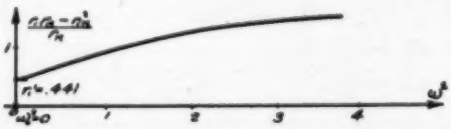


We then form:

$$\frac{r_1 r_n - r_{1n}^2}{r_n} =$$

$$= \frac{.23565\omega^8 - .03465\omega^6 + 1.00487\omega^4 - .13902\omega^2 + .25193}{.16356\omega^8 - .02025\omega^6 + .79662\omega^4 - .08060\omega^2 + .57165} \quad (11)$$

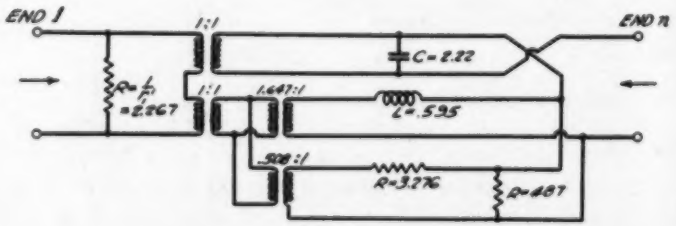
giving the curve:



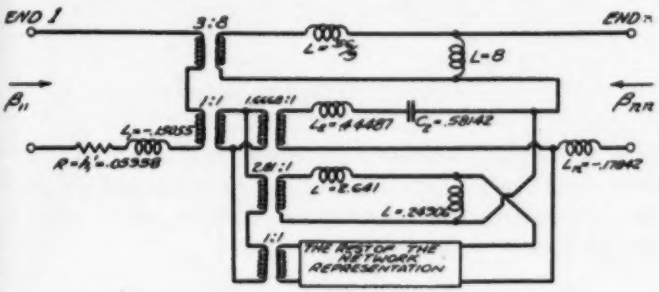
From this curve we see that its lowest point happens to fall at the origin (i.e. $\omega_0^2 = 0$) and thus r' is immediately found to be:

$$r_1' = .441$$

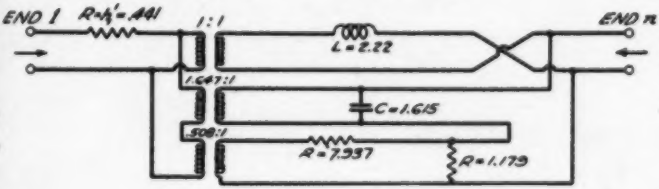
The imaginary parts for $\omega_0^2 = 0$ are all zero, *i.e. no "negative" reactance elements are introduced in this cycle*, and proceeding in the same manner as during the first cycle we find the network representation for this cycle to be, for instance, the one shown below:



On the other hand, had the prescribed functions been β:s the realization procedure would have been the same, but the realization for the first cycle would, for instance, have been:



and for the second cycle, the one shown below:



Example 24.

(a) Consider the elements of a "positive real" β-matrix:

$$\left.\begin{aligned} \beta_{11} &= \frac{2\lambda + 3}{2\lambda + 1} \\ \beta_{nn} &= \frac{\lambda + 2}{2\lambda + 1} \\ \beta_{1n} &= \frac{\lambda + 1}{2\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots (1)$$

From the location of the zeros and poles of all β:s and from the sequence of the zeros and the poles of β_{11} and β_{nn} we see that the above functions generally characterize a four-terminal network containing capacitance and resistance elements only.

Considering the $\frac{h_1 h_n - h_{1n}^2}{h_n} - \omega^2$ curve, say, shown below:

then, we see that, as required, its lowest point falls at ∞, and .5 units above the ω^2-axis. Thus, ω_0^2 is $= \infty$ and h_1' is $= .5$. Further, the imaginary parts of the prescribed β:s are all equal to zero at ∞, and we can write:

$$\left.\begin{aligned} \beta_{11} &= .5 + \frac{\lambda + 2.5}{2\lambda + 1} \\ \beta_{nn} &= \frac{\lambda + 2}{2\lambda + 1} \\ \beta_{1n} &= \frac{\lambda + 1}{2\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots(2)$$

Then, transforming above three fractions shown to the right into corresponding α:s we get:

$$\left.\begin{aligned} \alpha_{11} &= 8\lambda + \frac{1.8\lambda + 2}{2.5\lambda + 4} \\ \alpha_{nn} &= 8\lambda + \frac{2.8\lambda + 2.5}{2.5\lambda + 4} \\ \alpha_{1n} &= 8\lambda + \frac{-.2\lambda + 1}{2.5\lambda + 4} \end{aligned}\right\} \qquad \ldots\ldots(3)$$

Considering the $\frac{r_1 r_n - r_{1n}^2}{r_n} - \omega^2$ curve, say, we find, as required,

its lower point to fall at the origin, and .4 units above the ω^2-axis, as shown below:

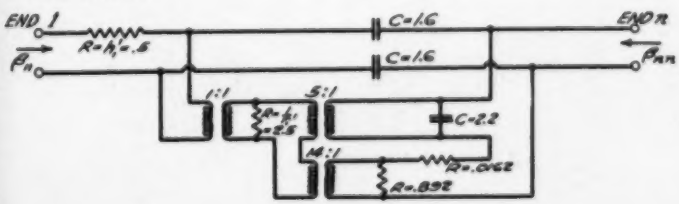
Thus, for this "half cycle" ω_0^2 is = 0 and r_1' is = .4, and we can write the component family to the right as:

$$\left.\begin{aligned} \alpha_{11} &= .4 + \frac{.8\lambda + .4}{2.5\lambda + 4} \\ \alpha_{nn} &= \frac{2.8\lambda + 2.5}{2.5\lambda + 4} \\ \alpha_{1n} &= \frac{-.2\lambda + 1}{2.5\lambda + 4} \end{aligned}\right\} \quad \ldots\ldots(4)$$

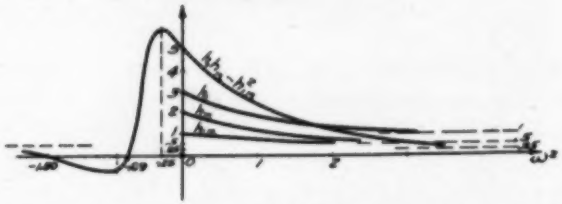
Then, transforming above three fractions into corresponding β:s, we get:

$$\left.\begin{aligned} \beta_{11} &= \frac{2.8\lambda + 2.5}{.88\lambda} = \frac{125}{44\lambda} + \frac{70}{22} \\ \beta_{nn} &= \frac{.8\lambda + .4}{.88\lambda} = \frac{20}{44\lambda} + \frac{20}{22} \\ \beta_{1n} &= \frac{-.2\lambda + 1}{.88\lambda} = \frac{50}{44\lambda} + \frac{5}{22} \end{aligned}\right\} \quad \ldots\ldots(5)$$

and the total network representation is, for instance, the one shown below:



(b) As mentioned, however, functions falling under item (g) Part X have the feature that the separate $(r_1 - \omega^2)$ and $r_n - \omega^2)$ curves or the separate $(h_1 - \omega^2)$ and $(h_n - \omega^2)$ curves have their lowest point at the same limiting frequency as where the corresponding determinant of the real parts $- \omega^2$ curves have their lowest point, as shown below.



making it possible to get a zero of the real parts of the two driving-point functions simultaneously with a zero of the proper determinant of the real parts.

$h_{1\min}$ and $h_{n\min}$ are readily found to be equal to 1.0 and .5 respectively, and the prescribed functions:

$$\left.\begin{aligned} \beta_{11} &= \frac{2\lambda + 3}{2\lambda + 1} \\ \beta_{nn} &= \frac{\lambda + 2}{2\lambda + 1} \\ \beta_{1n} &= \frac{\lambda + 1}{2\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots(1)$$

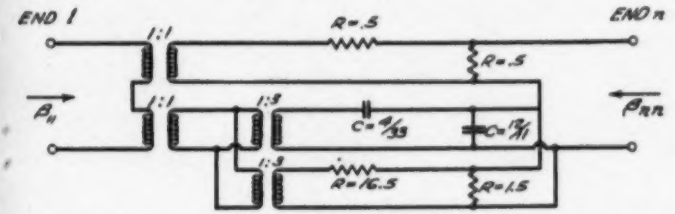
can thus be written:

$$\left.\begin{aligned} \beta_{11} &= 1.0 + \frac{2}{2\lambda + 1} \\ \beta_{nn} &= .5 + \frac{1.5}{2\lambda + 1} \\ \beta_{1n} &= .5 + \frac{.5}{2\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots(2)$$

Then, transforming the component family to the right into corresponding α:s, we get:

$$\left.\begin{aligned} \alpha_{11} &= \frac{3\lambda + 1.5}{2.75} = \frac{12}{11}\lambda + \frac{6}{11} \\ \alpha_{nn} &= \frac{4\lambda + 2}{2.75} = \frac{16}{11}\lambda + \frac{8}{11} \\ \alpha_{1n} &= \frac{\lambda + .5}{2.75} = \frac{4}{11}\lambda + \frac{2}{11} \end{aligned}\right\} \quad \ldots\ldots(3)$$

Note here, that the prescribed functions are exhausted of all their zeros and poles after one single inversion, and their network representation is, for instance, the one shown below:



If considering the prescribed functions as being α:s, instead of β:s, then the evaluation procedure would naturally be exactly the same as shown in the above two alternatives, although the network representations would be different.

Example 25. The quotient $\frac{r_1^2}{r_n}$ or $\frac{r_n^2}{r_1}$ can very well happen to be of the form:

$$f_{(\omega^2)} = \frac{\omega^4}{\omega^2 + 1} \quad \ldots\ldots(1)$$

The above function certainly is even and $\geq$ zero, for all real ω:s, but if considering it to be the real part of a rational frac-

tional function of λ, then this latter function, obtained by the aid of the algebraic method, is found to be:

$$f_{(\lambda)} = -\frac{\lambda^3 + \lambda^2 + \lambda}{\lambda + 1} \qquad \ldots\ldots(2)$$

which is *not* a "positive real" function because, firstly, all terms in one of the integral polynomials have the *wrong sign* and secondly, the power of λ in the numerator and the denominator *differs by more than one!*

Example 26. Let us consider a "positive real" α-matrix having the elements:

$$\left.\begin{aligned} \alpha_{11} &= \frac{\lambda^2 + \lambda}{\lambda^2 + 2\lambda + 2} \\ \alpha_{nn} &= \frac{\lambda^3 + .25\lambda + .5}{\lambda^2 + 2\lambda + 2} \\ \alpha_{1n} &= \frac{\lambda^2 + .5\lambda}{\lambda^2 + 2\lambda + 2} \end{aligned}\right\} \qquad \ldots\ldots(1)$$

Thus along the boundary we have:

$$\left.\begin{aligned} \alpha_{11(j\omega)} &= r_1 + jx_1 = \frac{\omega^4}{\omega^4 + 4} + j\frac{(- -)}{(- -)} \\ \alpha_{nn(j\omega)} &= r_n + jx_1 = \frac{\omega^4 - 2\omega^2 + 1}{\omega^4 + 4} + j\frac{(- -)}{(- -)} \\ \alpha_{1n(j\omega)} &= r_{1n} + jx_{1n} = \frac{\omega^4 - \omega^2}{\omega^4 + 4} + j\frac{(- -)}{(- -)} \end{aligned}\right\} \ldots(2)$$

The $(r - \omega^2)$ curves are plotted below:

From (2) we see that $(r_1 r_n - r_{1n}^2)$ is equal to zero, making it necessary to get $(\lambda^2 + 2\lambda + 2)$ as a factor in the numerator of the α-determinant.

This actually takes place, because:

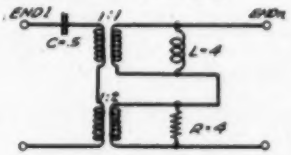
$$(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) = \frac{.25\lambda\,(\lambda^2 + 2\lambda + 2)}{(\lambda^2 + 2\lambda + 2)^2} \quad \ldots\ldots(3)$$

Let us continue and find a network representation of the α:s of (1).

After inversion, we get the β:s.

$$\left.\begin{aligned} \beta_{11} &= \frac{\lambda^2 + .25\lambda + .5}{.25\lambda} = \frac{2}{\lambda} + 4\lambda + 1 \\ \beta_{nn} &= \frac{\lambda^2 + \lambda}{.25\lambda} = 4\lambda + 4 \\ \beta_{1n} &= \frac{\lambda^2 + .5\lambda}{.25\lambda} = 4\lambda + 2 \end{aligned}\right\} \quad \ldots\ldots(4)$$

and the network representation is simply:



Example 27. Consider a "positive real" α-matrix having the elements:

$$\left.\begin{aligned} \alpha_{11} &= \frac{2\lambda^3 + 5\lambda^2 + 5\lambda + 1}{\lambda^2 + 2\lambda + 2} \\ \alpha_{nn} &= \frac{8\lambda^3 + 18\lambda^2 + 18\lambda + 1}{\lambda^2 + 2\lambda + 2} \\ \alpha_{1n} &= \frac{3\lambda^3 + 7\lambda^2 + 7\lambda - 1}{\lambda^2 + 2\lambda + 2} \end{aligned}\right\} \quad \ldots\ldots(1)$$

After removal of orginally existing poles on the boundary (here at ∞), we get:

$$\left.\begin{aligned} \alpha_{11} &= 2\lambda + \frac{\lambda^2 + \lambda + 1}{\lambda^2 + 2\lambda + 2} \\ \alpha_{nn} &= 8\lambda + \frac{2\lambda^2 + 2\lambda + 1}{\lambda^2 + 2\lambda + 2} \\ \alpha_{1n} &= 3\lambda + \frac{\lambda^2 + \lambda - 1}{\lambda^2 + 2\lambda + 2} \end{aligned}\right\} \quad \ldots\ldots (2)$$

The component family to the left can separately be realized, and letting the members of the component family to the right simply be α:s, we then get, when solving for their real and imginary parts along the boundary:

$$\left.\begin{aligned} \alpha_{11_{(j\omega)}} &= \frac{\omega^4 - \omega^2 + 2}{\omega^4 + 4} + j\omega \frac{\omega^2}{\omega^4 + 4} = r_1 + jx_1 \\ \alpha_{nn_{(j\omega)}} &= \frac{2\omega^4 - \omega^2 + 2}{\omega^4 + 4} + j\omega \frac{2\omega^2 + 2}{\omega^4 + 4} = r_n + jx_n \\ \alpha_{1n_{(j\omega)}} &= \frac{\omega^4 + \omega^2 - 2}{\omega^4 + 4} + j\omega \frac{\omega^2 + 4}{\omega^4 + 4} = r_{1n} + jx_{1n} \end{aligned}\right\} \ldots (3)$$

Making the extraction to the right, say, then we find the real part of Y_n, along the boundary, to be:

$$Y_{n,r} = \frac{r_1 r_n - r_{1n}^2}{r_1} = \frac{\omega^8 - 5\omega^6 + 10\omega^4}{(\omega^4 + 4)(\omega^4 - \omega^2 + 2)} \quad \ldots (4)$$

Then, the denominator of (4) is, by the aid of the expressions (159) and (160), found to generate the corresponding λ-expression:

$$(\lambda^2 + 2\lambda + 2)(\lambda^2 + 1.352193\lambda + 1.414213) \quad \ldots (5)$$

The proportionality factor is, from above λ-expression, the denominator of (4) and by the aid of (161),[235] found to be $= +1$, i.e. (5) is the denominator of Y_n (as well as of α'_{nn}).

[235] Note the b:s are the coefficients in the denominator of Y_n (or α'_{nn}), while those in the numerator are denoted as a:s.

Denoting the coefficients in the numerator of (4) with $A_4 A_3 \ldots A_0$, we then, by the aid of (162), get:

$$\left.\begin{aligned} A_0 &= 0 = a_0 b_0 \\ A_1 &= 0 = a_1 b_1 - (a_0 b_2 + a_2 b_0) \\ A_2 &= 10 = a_2 b_2 - (a_1 b_3 + a_3 b_1) + (a_0 b_4 + a_4 b_0) \\ A_3 &= -5 = a_3 b_3 - (a_2 b_4 + a_4 b_3) \\ A_4 &= 1 = a_4 b_4 \end{aligned}\right\} \ldots (6)$$

The solution of (6) gives:

$$\left.\begin{aligned} a_0 &= 0 \\ a_1 &= 1.673668 \\ a_2 &= 3.273942 \\ a_3 &= 1.310349 \\ a_4 &= 1.0 \end{aligned}\right\} \qquad \ldots\ldots (7)$$

and thus we have our Y_a as:

$$Y_a = \frac{\lambda^4 + 1.310349\lambda^3 + 3.273942\lambda^2 + 1.673668\lambda}{\lambda^4 + 3.352193\lambda^3 + 6.118600\lambda^2 + 5.532814\lambda + 2.828427} \ldots (8)$$

This is readily found to be a "positive real" function and it can be realized by known methods.[236]

Then, α'_{aa} can be obtained from:

$$\alpha'_{aa} = \alpha_{aa} - Y_a$$

and is found to be:

$$\alpha'_{aa} = \frac{\lambda^4 + 3.394037\lambda^3 + 3.258871\lambda^2 + 2.506952\lambda + 1.414213}{(\lambda^2 + 2\lambda + 2)(\lambda^2 + 1.352193\lambda + 1.414213)} \ldots (9)$$

As a check, let r'_a generate α'_{aa} in the same way as Y_{ar} generated Y_a.

[236] Bibliogr. 38.

We have:

$$r_n' = \frac{r_{1n}^2}{r_1} = \frac{\omega^8 + 2\omega^6 - 3\omega^4 - 4\omega^2 + 4}{(\omega^4 + 4)(\omega^4 - \omega^2 + 2)} \quad \ldots (10)$$

The denominator of α_{nn}' is naturally (5) and, by the aid of (162), we get:

$$\left.\begin{aligned} A_0 &= 4 = a_0 b_0 \\ A_1 &= -4 = a_1 b_1 - (a_0 b_2 + a_2 b_0) \\ A_2 &= -3 = a_2 b_2 - (a_1 b_3 + a_3 b_1) + (a_0 b_4 + a_4 b_0) \\ A_3 &= 2 = a_3 b_3 - (a_2 b_4 + a_4 b_2) \\ A_4 &= 1 = a_4 b_4 \end{aligned}\right\} \ldots (11)$$

The solution of (11) gives:

$$\left.\begin{aligned} a_0 &= 1.414213 \\ a_1 &= 2.506949 \\ a_2 &= 3.258871 \\ a_3 &= 3.394037 \\ a_4 &= 1.0 \end{aligned}\right\} \quad \ldots\ldots (12)$$

and we get our α_{nn}':

$$\alpha_{nn}' = \frac{\lambda^4 + 3.394037\lambda^3 + 3.258871\lambda^2 + 2.506949\lambda + 1.414213}{(\lambda^2 + 2\lambda + 2)(\lambda^2 + 1.352193\lambda + 1.414213)} \ldots (13)$$

As required, we have practically a "cold" check between (9) and (13).

Further, note that the power of λ in Y_n and α_{nn}' is twice as high as in α_{nn}, which is correct, and if adding Y_n and α_{nn}' they (as they should) give α_{nn}.

When now forming the determinant $(\alpha_{11}\,\alpha_{nn}' - \alpha_{1n}^2)$ its numerator should be of the fifth degree and its denominator of the sixth degree in λ. Further, λ and $(\lambda^2 + 2\lambda + 2)$ should become factors in the numerator.

This actually happens as:

$$(\alpha_{11}\alpha'_{nn} - \alpha_{1n}^2) =$$

$$\frac{\lambda(1.041841\lambda^4 + 4.534298\lambda^3 + 9.683608\lambda^2 + 10.298620\lambda + 5.397392)}{(\lambda^2 + 2\lambda + 2)^2 (\lambda^2 + 1.352193\lambda + 1.414213)}$$

$$= \frac{\lambda(\lambda^2 + 2\lambda + 2)\,(1.041841\lambda^2 + 2.450616\lambda + 2.698695)}{(\lambda^2 + 2\lambda + 2)^2 (\lambda^2 + 1.352193\lambda + 1.414213)} \quad \ldots\ldots (14)$$

When forming the β:s, two (at least) poles on the boundary should "fall out," that is, can be removed from the boundary. Then, what remains should, at most, be of the second degree in λ (in the numerator as well as in the denominator).

Further, the determinant of the residues of the removed poles should be = 0.

We get:

$$\left.\begin{aligned}
\beta_{11} &= \frac{\lambda^4 + 3.394037\lambda^3 + 3.258871\lambda^2 + 2.506950\lambda + 1.414213}{\lambda(1.041841\lambda^2 + 2.450616\lambda + 2.698695)} = \\
&\frac{.524035}{\lambda} + .959839\lambda + \frac{1.041841\lambda^2 + .122598\lambda + 1.222742}{1.041841\lambda^2 + 2.450616\lambda + 2.698695} \\
\beta_{nn} &= \frac{(\lambda^2 + \lambda + 1)\,(\lambda^2 + 1.352193\lambda + 1.414213)}{\lambda(1.041841\lambda^2 + 2.450616\lambda + 2.698695)} = \\
&\frac{\lambda^4 + 2.352193\lambda^3 + 3.766406\lambda^2 + 2.766406\lambda + 1.414213}{\lambda(1.041841\lambda^2 + 2.450616\lambda + 2.698695)} = \\
&\frac{.524035}{\lambda} + .959839\lambda + \frac{.630132\lambda + 1.482198}{1.041841\lambda^2 + 2.450616\lambda + 2.698695} \\
\beta_{1n} &= \frac{(\lambda^2 + \lambda - 1)\,(\lambda^2 + 1.352193\lambda + 1.414213)}{\lambda(1.041841\lambda^2 + 2.450616\lambda + 2.698695)} = \\
&\frac{\lambda^4 + 2.352193\lambda^3 + 1.766406\lambda^2 + 0.062020\lambda - 1.414213}{\lambda(1.041841\lambda^2 + 2.450616\lambda + 2.698695)} = \\
&\frac{-.524035}{\lambda} + .959839\lambda + \frac{-.277945\lambda + 1.346228}{1.041841\lambda^2 + 2.450616\lambda + 2.698695}
\end{aligned}\right\} . (15)$$

The poles on the boundary, removed from the β:s, do not change their real parts, which are:

$$\left.\begin{aligned} h_1 &= \frac{1.085432\omega^4 - 3.785073\omega^2 + 3.299807}{1.085432\omega^4 + .382296\omega^2 + 7.282954} \\ h_n &= \frac{4.000000}{1.085432\omega^4 + .382296\omega^2 + 7.282954} \\ h_{1n} &= \frac{-2.083691\omega^2 + 3.633058}{1.085432\omega^4 + .382296\omega^2 + 7.282954} \end{aligned}\right\} \quad \ldots (16)$$

Then, the determinant of the h:s of (16) should be = 0!

$$(h_1 h_n - h_{1n}^2) =$$

$$\frac{4(1.085432\omega^4 - 3.785073\omega^2 + 3.299807) - (-2.083691\omega^2 + 3.633058)^2}{(1.085432\omega^4 + .382296\omega^2 + 1.282954)^2} . \quad (17)$$

i.e. it *is* = 0.

Denoting the β:s of (15), after the two poles have "fallen out," with β'':s, and forming the β''-determinant, then λ and $(1.041841\lambda^2 + 2.450616\lambda + 2.698695)$ should become factors in its numerator.

$$(\beta''_{11}\beta''_{nn} - \beta''^2_{1n}) = \frac{\lambda(.656498\lambda^2 + 1.544214\lambda + 1.700558)}{(1.041841\lambda^2 + 2.450616\lambda + 2.698695)^2} =$$

$$\frac{.630132\lambda(1.041841\lambda^2 + 2.450616\lambda + 2.698695)}{(1.041841\lambda^2 + 2.450616\lambda + 2.698695)^2} . \quad (18)$$

When inverting the β'':s into corresponding α'':s, two poles (if as many are left) should again "fall out" from each α'', leaving a rest *which should be a real constant!* The determinant of the residues at the poles, removed from the boundary, should again be = 0, and *in addition the determinant of said constants should also be* = 0, due to the requirement of the determinant of the real parts to be = 0.

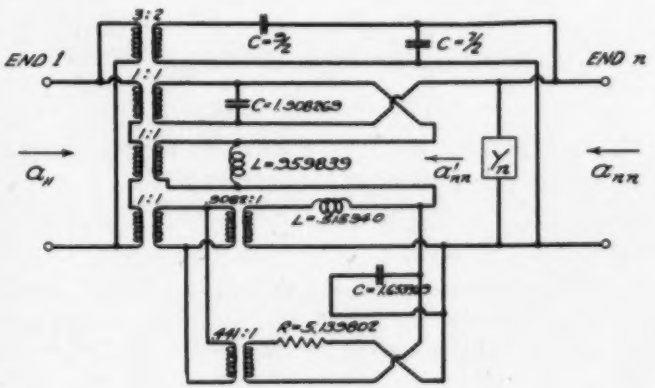
The α'''s are found to be:

$$\left.\begin{aligned} \alpha''_{11} &= \frac{.630132\lambda + 1.482198}{.630132\lambda} \\ \alpha''_{nn} &= \frac{1.041841\lambda^2 + .122598\lambda + 1.222742}{.630132\lambda} \\ \alpha''_{1n} &= \frac{-.277945\lambda + 1.346228}{.630132\lambda} \end{aligned}\right\} \quad \ldots\ldots(19)$$

i.e.

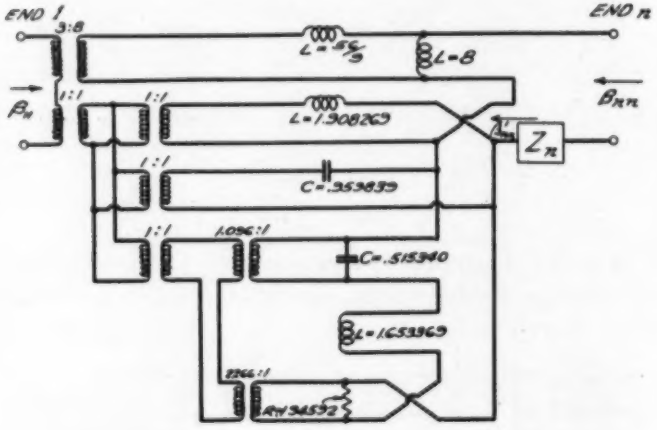
$$\left.\begin{aligned} \alpha''_{11} &= \frac{2.352202}{\lambda} + 1.0 \\ \alpha''_{nn} &= 1.653369\lambda + \frac{1.940453}{\lambda} + .194559 \\ \alpha''_{1n} &= \frac{2.136422}{\lambda} - .441090 \end{aligned}\right\} \quad \ldots\ldots(20)$$

Thus the prescribed functions exactly followed predicted requirements, and the network representation is, for instance, as shown below.



Y_n is given by (8) and can be realized into a dissipative two-terminal network having at most eight elements.

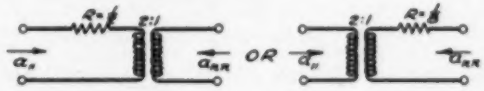
On the other hand, had the prescribed functions, as given by (1), been β:s, then we would have to give the "extracted" "positive real" function (8) an interpretation as an impedance. As to the rest, the evaluation procedure of the prescribed matrix would be exactly analogous to the one given in this Example, although the network representation in case of a prescribed β-matrix would, for instance, be as shown below.



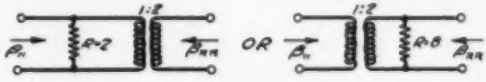
Example 28. Suppose the following α:s are prescribed, or they happen to be members of a component family:

$$\left.\begin{aligned} \alpha_{11} &= 2 \\ \alpha_{nn} &= 8 \\ \alpha_{1n} &= 4 \end{aligned}\right\} \qquad \ldots\ldots(1)$$

Thus $(\alpha_{11}\alpha_{nn} - \alpha_{1n}^2) = 0$, and the network representation is simply:



On the other hand, in case the constants in (1) are β:s, then the network representation is:



Example 29. Suppose a singular "positive real" α-matrix has the elements:

$$\left.\begin{aligned} \alpha_{11} &= \frac{4\lambda}{\lambda + 1} \\ \alpha_{nn} &= \frac{\lambda}{\lambda + 1} \\ \alpha_{1n} &= \frac{2\lambda}{\lambda + 1} \end{aligned}\right\} \qquad \ldots\ldots (1)$$

Thus, along the boundary, we get:

$$\left.\begin{aligned} \alpha_{11(j\omega)} &= r_1 + jx_1 = \frac{4\omega^2}{\omega^2 + 1} + j\omega\,\frac{4}{\omega^2 + 1} \\ \alpha_{nn(j\omega)} &= r_n + jx_n = \frac{\omega^2}{\omega^2 + 1} + j\omega\,\frac{1}{\omega^2 + 1} \\ \alpha_{1n(j\omega)} &= r_{1n} + jx_{1n} = \frac{2\omega^2}{\omega^2 + 1} + j\omega\,\frac{2}{\omega^2 + 1} \end{aligned}\right\} \ldots\ldots (2)$$

We know that the r-determinant is zero, and in addition we get:

$$(r_{1n}\, x_n - r_n\, x_{1n}) = \frac{2\omega^3 - 2\omega^3}{(\omega^2 + 1)^2} \qquad \ldots\ldots (3)$$

i.e. also this expression is zero. Thus, we see that the ratio (202) of the transformer is a real constant!

The two-terminal network, $Z = \dfrac{1}{\alpha_{11}}$ in Fig. 56, is in this case:

$$Z = \frac{\lambda + 1}{4\lambda} = \frac{1}{4\lambda} + \frac{1}{4} \qquad \ldots\ldots (4)$$

i.e. the network representation of the functions in (1) is, for instance:

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FARADAY'S LAW AS A BASIS OF ELECTROMAGNETIC THEORY

By H. B. Phillips

1. *Introduction.* An electromagnetic field is characterized by electric intensity **E** and magnetic induction **B**. According to Faraday's law of electromagnetic induction, the line integral of **E** around any closed curve is proportional to the rate of diminution of magnetic flux through a surface having that curve as boundary. This and other principles are usually combined in the development of electromagnetic theory. In this paper I wish to show that with any two vectors satisfying Faraday's law are associated quantities having the properties of charges, currents, and potentials. Any such pair of vectors thus determine a possible electromagnetic field.

2. *Faraday's Law.* Let **R** be the vector from a fixed origin to the point $P(x,y,z)$, **n** the unit normal, and da the scalar element of area on a surface S. Faraday's law is then expressed by the equation

$$\int_C \mathbf{E}\cdot d\mathbf{R} = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot\mathbf{n}\, da, \tag{1}$$

where C is the curve forming the boundary of S. In this equation t is the time, c is the velocity of light, and the directions of $d\mathbf{R}$ and **n** are related as in Stokes's theorem.

For this equation to hold it is necessary that the surface integral depend only on the boundary and not otherwise on the surface S. Consequently

$$\frac{\partial}{\partial t}\oint \mathbf{B}\cdot\mathbf{n}\, da = 0,$$

the circle on the integral sign indicating that the surface is closed. We then have

$$\oint \mathbf{B}\cdot\mathbf{n}\, da = k,$$

where k may depend on the surface but not on the time. If the field is one which can be created or destroyed, this equation will be valid before the values of $\mathbf{B}$ have come into existence or after they have disappeared. Hence the constant is zero, that is,

$$\oint \mathbf{B}\cdot\mathbf{n}\, da = 0. \tag{2}$$

Since this is true for every closed surface,

$$\nabla\cdot\mathbf{B} = 0 \tag{3}$$

at each point where $\mathbf{B}$ has continuous first derivatives with respect to x, y, z.

On a surface S where $\mathbf{E}$ and its first derivatives with respect to x, y, z are continuous, by Stokes's theorem equation (1) is equivalent to

$$\int_S \mathbf{n}\cdot\nabla\times\mathbf{E}\, da = -\frac{1}{c}\int_S \mathbf{n}\cdot\frac{\partial\mathbf{B}}{\partial t}\, da.$$

Since this is true for every surface S,

$$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0 \tag{4}$$

at each point where the derivatives used are continuous.

There may be certain regular surfaces on each side of which $\mathbf{E}$, $\mathbf{B}$, $\dfrac{\partial\mathbf{B}}{\partial t}$ are continuous, but on which they have finite discontinuities. Let $\Delta\mathbf{E}$, $\Delta\mathbf{B}$ be the increments in $\mathbf{E}$ and $\mathbf{B}$ when the point at which these quantities are measured crosses such a surface in the positive direction of the normal. Equations (1) and

(2) show that on such a surface the tangential component of $\mathbf{E}$ and the normal component of $\mathbf{B}$ are continuous. That is,

$$\mathbf{n} \times \Delta\mathbf{E} = 0, \tag{5}$$

$$\mathbf{n} \cdot \Delta\mathbf{B} = 0. \tag{6}$$

Equations (3) to (6) express Faraday's law as conditions satisfied at a point. Since however $\mathbf{B}$ and $\mathbf{E}$ may not at all points have the simple character assumed in these equations we can give the theory a more general foundation if instead of these vectors we use two quantities $\mathbf{U}$ and $\mathbf{V}$ which we now proceed to define.

3. *The Vectors* $\mathbf{U}$ *and* $\mathbf{V}$. With a point $P(x,y,z)$ and time t we associate two integrals

$$\mathbf{U} = \int \frac{\mathbf{B}'\, dv}{r}, \tag{7}$$

$$\mathbf{V} = \int \frac{\mathbf{E}'\, dv}{r}, \tag{8}$$

taken over all space. In these dv is the element of volume at $P'(x',y',z')$, r is the distance PP' and

$$\mathbf{B}' = \mathbf{B}(x', y', z', t'),$$

$$\mathbf{E}' = \mathbf{E}(x', y', z', t')$$

are values of $\mathbf{B}$ and $\mathbf{E}$ at P' at time

$$t' = t - \frac{r}{c}.$$

I shall assume that $\mathbf{B}$, $\mathbf{E}$, and $\dfrac{\partial \mathbf{B}}{\partial t}$ are bounded. By this it is meant that a fixed number exists which exceeds these vectors in magnitude at every point of the field. While mathematical expressions are sometimes used which give for these vectors infi-

nite values at certain points, it is doubtful if such expressions are more than approximations, the real vectors being everywhere finite. Since the limits of integration are infinite it is necessary however to prove that in cases of physical interest the integrals **U** and **V** converge.

To show this we note first that although the field may extend to infinity the charges and currents which give rise to it are always located in a limited region. Let the point P at which **U** and **V** are measured be restricted to a finite region Γ containing the origin and all charges, currents, polarized and magnetized material. Also unless the field is static it loses energy. If then we go back far enough we always come to a time when the field was static. As we go out from the origin the time t' steadily diminishes. There will then be a sphere S_1 with center at the origin, containing Γ, outside which $\mathbf{E}'$ and $\mathbf{B}'$ have the values of **E** and **B** in a static field whatever point P is taken in Γ. In such a field **E** and **B** are gradients of harmonic functions. To prove the convergence of (7) and (8) it will then be sufficient to show that

$$\int \frac{\nabla\phi}{r}\, dv$$

converges when ϕ is harmonic and the integral is taken over the region outside a sphere.

We have

$$\int \frac{\nabla\phi}{r}\, dv = \int \frac{\nabla\phi}{r_0}\, dv + \int \left(\frac{1}{r} - \frac{1}{r_0}\right) \nabla\phi\, dv, \tag{9}$$

where r_0 is the distance between dv and the origin. Since ϕ is harmonic outside S_1 there is a number m such that

$$|r_0^2\, \nabla\phi| < m \tag{10}$$

at all points of that region. Since P belongs to a region within S_1,

$$r_0^2\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

is also bounded for points P' outside S_1. The second integral on the right of (9) therefore converges absolutely. The quantity $\nabla\phi$ is equivalent to the electric intensity due to a distribution of charges within S_1. Its average value over an outside sphere S is therefore zero. That is,

$$\int_S \nabla\phi \, da = 0.$$

Thus if we integrate over the region inside a sphere with center at the origin and then let the radius of that sphere increase indefinitely we get

$$\int \frac{\nabla\phi}{r_0} dv = 0$$

for the integral over the region outside S_1. The first integral in (9) when calculated in this way thus also converges.

But this integral may not converge absolutely and so its value may depend on the way the integration is performed. From (10) the part of the integral between two spheres of radii r_0, $r_0 + h$ is however less than

$$\int_{r_0}^{r_0+h} \frac{4\pi m \, dr_0}{r_0} = 4\pi m \ln\left(\frac{r_0 + h}{r_0}\right)$$

which approaches zero when r_0 is indefinitely increased. If instead of the sphere S we integrate over the interior of any regular closed surface Σ and then allow Σ to increase indefinitely, we shall therefore get the same limit provided Σ at each stage of the process lies between two spheres with centers at the origin and radii differing by not more than a fixed amount h. In particular we may use spheres with any fixed point as center. With this understanding as to the process of summation the integrals (7) and (8) then converge and for points P in a limited region the convergence is uniform.

4. *Faraday's Law in Terms of* **U** *and* **V**. We shall show that if **B** and **E** satisfy equation (1), **U** and **V** satisfy the same equation. In terms of **U** and **V** Faraday's law thus takes the form

$$\int_C \mathbf{V}\cdot d\mathbf{R} = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \mathbf{U}\cdot\mathbf{n}\, da. \tag{11}$$

This will appear as a consequence of certain relations which will first be obtained.

Let $Q(\xi, \eta, \zeta)$ be any definite point associated with the surface S. With any other point Q' (ξ', η', ζ') of space associate the surface S' of boundary C' obtained by giving to each point $P(x,y,z)$ on S the vector displacement

$$\overrightarrow{PP'} = \overrightarrow{QQ'} \tag{12}$$

Consider the quantities

$$\lambda'(Q') = \int_{C'} \mathbf{E}'(P')\cdot d\mathbf{R} + \frac{1}{c}\int_{S'} \mathbf{n}\cdot\frac{\partial \mathbf{B}'(P')}{\partial t'}\, da, \tag{13}$$

$$\mu\,(Q) = \int_C \mathbf{V}(P)\cdot d\mathbf{R} + \frac{1}{c}\int_S \mathbf{n}\cdot\frac{\partial \mathbf{U}(P)}{\partial t}\, da, \tag{14}$$

where $\mathbf{U}(P)$, $\mathbf{V}(P)$ are values of **U**, **V** at P at time t and $\mathbf{B}'(P')$, $\mathbf{E}'(P')$ are values of **B** and **E** at P' at time

$$t' = t - \frac{r}{c}$$

and r is the distance $PP' = QQ'$. Not assuming that **B** and **E** satisfy Faraday's law but merely that they are any vectors which are bounded and have the properties we have assumed at infinity, we shall first prove that

$$\mu(Q) = \int \frac{\lambda'(Q')\, dv}{r} \tag{15}$$

the integral being taken over all space.

Since $\mathbf{E}$ and $\mathbf{B}$ at a sufficiently great distance have the values belonging to a static field, each integral on the right of (13) is zero when r is sufficiently large. The integral on the right of (15) therefore converges. Replacing $\lambda'(Q')$ by its value we obtain

$$\int \frac{\lambda'(Q')\,dv}{r} = \int \frac{dv}{r}\left[\int_C \mathbf{E}'(P')\cdot d\mathbf{R} + \frac{1}{c}\int_{S'} \mathbf{n}\cdot\frac{\partial \mathbf{B}'(P')}{\partial t'}\,da\right]. \quad (16)$$

The surface S' being obtained by parallel displacement from the surface S the elements of integration $d\mathbf{R}$ and $\mathbf{n}\,da$ may be taken to have the same values at P and P'. We could therefore consider

$$\int_C \mathbf{E}'(P')\cdot d\mathbf{R}, \qquad \int_{S'} \mathbf{n}\cdot\frac{\partial \mathbf{B}'(P')}{\partial t'}\,da$$

as evaluated on S, the element of integration at each point P being multiplied by the value of $\mathbf{E}'$ or $\dfrac{\partial \mathbf{B}'}{\partial t'}$ at P'. Inverting the order of integration (16) can thus be written

$$\int \frac{\lambda'(Q')\,dv}{r} = \int_C d\mathbf{R}\cdot\int \frac{\mathbf{E}'(P')\,dv}{r} + \frac{1}{c}\int_S \mathbf{n}\,da\cdot\int \frac{1}{r}\frac{\partial \mathbf{B}'(P')}{\partial t'}\,dv. \quad (17)$$

This inversion is justified because the parts of (16) and (17) for which the integrand is not bounded belong to the region $r < \epsilon$ and these parts can be made as small as we please by taking ϵ sufficiently small. In these integrals dv is located at Q' and r is the distance QQ'. If we give to all these elements of volume the vector displacement

$$\overrightarrow{QP} = \overrightarrow{Q'P'}$$

r becomes the distance PP' and dv is moved to P'. The integrals

$$\int \frac{\mathbf{E}'(P')\,dv}{r}, \qquad \int \frac{\mathbf{B}'(P')\,dv}{r}$$

are thus the values of $\mathbf{V}$ and $\mathbf{U}$ at P and so (17) is equivalent to

$$\int \frac{\lambda'(Q')\,dv}{r} = \int_C \mathbf{V}\cdot d\mathbf{R} + \frac{1}{c}\frac{\partial}{\partial t}\int_S \mathbf{U}\cdot\mathbf{n}\,da \tag{18}$$

which is equation (15). If now we assume that $\mathbf{B}$ and $\mathbf{E}$ satisfy Faraday's law, $\lambda'(Q')$ is zero for all positions of Q' and (18) gives (11) which was to be proved.

Conversely, if equation (11) is satisfied equation (1) is also, at least in regions where $\mathbf{E}$ and $\mathbf{B}$ have continuous first derivatives with respect to x, y, z and continuous second derivatives with respect to t. For then

$$\mu = \int \frac{\lambda'(Q')\,dv}{r} = 0$$

being a retarded potential, satisfies d'Alembert's equation and so

$$4\pi\lambda(Q) = \frac{1}{c^2}\frac{\partial^2\mu}{\partial t^2} - \nabla^2\mu = 0.$$

Thus (11) may be taken as completely representing Faraday's law.

Since $\mathbf{U}$ and $\mathbf{V}$ are continuous and have continuous first derivatives with respect to x, y, z from equation (11) we conclude that

$$\nabla\cdot\mathbf{U} = 0, \tag{19}$$

$$\nabla\times\mathbf{V} + \frac{1}{c}\frac{\partial\mathbf{U}}{\partial t} = 0, \tag{20}$$

the argument being identical with that used in § 2 to derive the analogous equations (3) and (4) from equation (1).

5. *Scalar and Vector Potential.* With each point of space and time t associate the complex quantity

$$\mathbf{W} = \mathbf{U} + i\mathbf{V} = \int \frac{\mathbf{B}' + i\mathbf{E}'}{r}\,dv \tag{21}$$

where $i = \sqrt{-1}$. This quantity and its first derivatives are everywhere continuous. Let ϕ be the scalar and $\mathbf{A}$ the vector defined by the equations

$$4\pi i \phi = \nabla \cdot \mathbf{W}, \tag{22}$$

$$4\pi \mathbf{A} = \nabla \times \mathbf{W} + \frac{i}{c} \frac{\partial \mathbf{W}}{\partial t}. \tag{23}$$

These are called the scalar and the vector potential. Equations (19) and (20) show that they are real. The effect of Faraday's law is thus merely to make the potentials real.

From (22) and (23) by differentiating and substituting we find

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0. \tag{24}$$

The significance of this equation will be considered in § 7.

At points where $\mathbf{B}$ and $\mathbf{E}$ have continuous first derivatives with respect to x, y, z and continuous second derivatives with respect to t, each component of $\mathbf{W}$ satisfies d'Alembert's equation and so $\mathbf{W}$ itself satisfies the equation

$$\nabla^2 \mathbf{W} - \frac{1}{c^2} \frac{\partial^2 \mathbf{W}}{\partial t^2} = -4\pi(\mathbf{B} + i\mathbf{E}). \tag{25}$$

From (23) we have

$$\nabla \times \mathbf{W} = 4\pi \mathbf{A} - \frac{i}{c} \frac{\partial \mathbf{W}}{\partial t},$$

whence

$$\frac{i}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{W} = \frac{4\pi}{c} \frac{i}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{W}}{\partial t^2}.$$

Also

$$4\pi \nabla \times \mathbf{A} = \nabla \times (\nabla \times \mathbf{W}) + \frac{i}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{W},$$

$$\nabla \times (\nabla \times \mathbf{W}) = \nabla (\nabla \cdot \mathbf{W}) - \nabla^2 \mathbf{W}.$$

Adding the last three equations and subtracting (25) we can eliminate $\mathbf{W}$ and so obtain

$$\mathbf{B} + i\mathbf{E} = \nabla \times \mathbf{A} - i\left(\nabla \phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right). \tag{26}$$

Since $\mathbf{B}$, $\mathbf{E}$, $\mathbf{A}$, and ϕ are real,

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{27}$$

$$\mathbf{E} = -\nabla \phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{28}$$

These are the usual expressions for magnetic induction and electric intensity in terms of the potentials.

6. *Field in a Material Medium.* To advance further we must specify the nature of the medium. This is usually done by means of the dielectric constant and the permeability. We shall not introduce these quantities but merely assume that $\mathbf{D}$, $\mathbf{P}$, $\mathbf{H}$, $\mathbf{M}$ are vectors which satisfy the equations

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \tag{29}$$

$$\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M} \tag{30}$$

and that $\mathbf{P}$ and $\mathbf{M}$ have values different from zero only in a finite region. The vector $\mathbf{D}$ represents electric induction, $\mathbf{H}$ magnetic force, $\mathbf{P}$ and $\mathbf{M}$ electric and magnetic polarization. If functions μ and k exist such that

$$\mathbf{D} = k\mathbf{E}, \qquad \mathbf{B} = \mu\mathbf{H}$$

we can express all these quantities in terms of $\mathbf{B}$ and $\mathbf{E}$ but there will be no occasion to use such relations.

At points where $\mathbf{D}$ and $\mathbf{H}$ have continuous first derivatives with respect to x, y, z let ρ and $\mathbf{I}$ be defined by the equations

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \tag{31}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\left(4\pi \mathbf{I} + \frac{\partial \mathbf{D}}{\partial t}\right). \tag{32}$$

On a surface where $\mathbf{D}(x, y, z, t)$, $\mathbf{H}(x, y, z, t)$ have finite increments $\Delta\mathbf{D}$, $\Delta\mathbf{H}$ when the point $P(x, y, z)$ crosses the surface in the positive direction of the normal, let σ and $\mathbf{J}$ be defined by the equations

$$\mathbf{n}\cdot\Delta\,\mathbf{D} = 4\,\pi\,\sigma, \tag{33}$$

$$\mathbf{n} \times \Delta\,\mathbf{H} = \frac{4\,\pi\,\mathbf{J}}{c}. \tag{34}$$

The scalars ρ and σ represent space and surface charge, the vectors $\mathbf{I}$ and $\mathbf{J}$ space and surface current.

To obtain the potentials we must find the divergence and curl of

$$\mathbf{U}(x, y, z, t) = \int \frac{\mathbf{B}'\,dv}{r} = \int \frac{\mathbf{H}' + 4\,\pi\,\mathbf{M}'}{r}\,dv, \tag{35}$$

$$\mathbf{V}(x, y, z, t) = \int \frac{\mathbf{E}'\,dv}{r} = \int \frac{\mathbf{D}' - 4\,\pi\,\mathbf{P}'}{r}\,dv. \tag{36}$$

In these integrals r is the distance between $P(x, y, z)$ and the point $P'(x', y', z')$ at the element of integration. Quantities not under integral signs are functions of x, y, z, t. Applied to such quantities we use the notation

$$\nabla = \nabla_P = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

to represent differentiation with respect to the coordinates of P with t constant. Under the integral signs the variables are x', y', z', t' and we use the notation

$$\nabla = \nabla_{P'} = \mathbf{i}\,\frac{\partial}{\partial x'} + \mathbf{j}\,\frac{\partial}{\partial y'} + \mathbf{k}\,\frac{\partial}{\partial z'}$$

to represent differentiation with respect to the coordinates of P' with t' constant. In case of a function of r we obviously have

$$\nabla_P\,f(r) = -\,\nabla_{P'}\,f(r).$$

The integrals are functions of x, y, z through the quantity r which appears not only in the denominators but also in the expression for t'. Thus

$$\nabla \cdot \int \frac{\mathbf{D}'}{r}\, dv = -\int \left[\mathbf{D}' \cdot \nabla \frac{1}{r} - \frac{1}{rc} \frac{\partial \mathbf{D}'}{\partial t'} \cdot \nabla r \right] dv,$$

the negative sign being due to the change under the integral sign from differentiation with respect to the coordinates of P to differentiation with respect to those of P'. The right side of this equation is equivalent to

$$\int \left[\frac{\nabla \cdot \mathbf{D}'}{r} - \nabla_t \cdot \frac{\mathbf{D}'}{r} \right] dv,$$

where ∇_t signifies differentiation with respect to the coordinates of P' with t constant. By the divergence theorem

$$\int \nabla_t \cdot \frac{\mathbf{D}'}{r}\, dv = \int \frac{\mathbf{n} \cdot \mathbf{D}'\, da}{r},$$

the integral on the right being extended over both sides of any surface on which $\mathbf{D}'$ has discontinuities. By equation (33) we thus have

$$\int \nabla_t \cdot \frac{\mathbf{D}'}{r}\, dv = -4\pi \int \frac{\sigma'\, da}{r}.$$

Combining these equations and replacing $\nabla \cdot \mathbf{D}'$ by means of (31) we finally obtain

$$\nabla \cdot \int \frac{\mathbf{D}'}{r}\, dv = 4\pi \int \frac{\rho'\, dv}{r} + 4\pi \int \frac{\sigma'\, da}{r}. \tag{37}$$

From equations (22), (36), and (37) we have

$$\phi = \int \frac{\rho'\, dv}{r} + \int \frac{\sigma'\, da}{r} - \nabla \cdot \int \frac{\mathbf{P}'\, dv}{r}. \tag{38}$$

Proceeding in a similar way from (23) we obtain

$$\mathbf{A} = \int \frac{1}{cr}\left(\mathbf{I}' + \frac{\partial \mathbf{P}'}{\partial t'}\right) dv + \int \frac{\mathbf{J}'\, da}{cr} + \nabla \times \int \frac{\mathbf{M}'\, dv}{r}. \quad (39)$$

These equations represent the potentials of a distribution which has only surface discontinuities. The integrals are extended over all volumes and surfaces where the respective integrands have values.

7. *Equation of Continuity.* The two potentials satisfy equation (24), namely

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0. \quad (40)$$

In a variable field we may consider the scalar potential as flowing from point to point. This equation expresses that the vector potential is the current of scalar potential.

If in (40) we substitute the values of ϕ and $\mathbf{A}$ from (38) and (39) the terms containing $\mathbf{P}'$ and $\mathbf{M}'$ vanish. After the usual integral transformations the remainder becomes

$$\int \frac{1}{r}\left(\nabla \cdot \mathbf{I}' + \frac{\partial \rho'}{\partial t}\right) dv + \int \frac{1}{r}\left(\mathbf{n} \cdot \Delta \mathbf{I}' + \nabla \cdot \mathbf{J}' + \frac{\partial \sigma'}{\partial t'}\right) da = 0.$$

In this equation $\Delta \mathbf{I}'$ is the increment in $\mathbf{I}'(x', y', z', t')$ when the point $P'(x', y', z')$ crosses a surface on which $\mathbf{I}'$ is discontinuous in the positive direction of the normal $\mathbf{n}$, and $\nabla \cdot \mathbf{J}'$ is the surface divergence of $\mathbf{J}'$. If this equation is satisfied everywhere we must have

$$\nabla \cdot \mathbf{I} + \frac{\partial \rho}{\partial t} = 0, \quad (41)$$

$$\mathbf{n} \cdot \Delta \mathbf{I} + \nabla \cdot \mathbf{J} + \frac{\partial \sigma}{\partial t} = 0. \quad (42)$$

These equations express the continuity of electric charge, the first at a point in space, the second on a surface of discontinuity.

8. *Uniqueness.* Starting with two vectors $\mathbf{B}$ and $\mathbf{E}$ which satisfy Faraday's law and assuming arbitrary values of $\mathbf{P}$ and $\mathbf{M}$, we have defined a series of quantities connected by equations of the same form as those satisfied by charges, currents, and potentials in electromagnetic theory. The potentials are continuous everywhere with limits zero at infinity, are differentiable except on certain surfaces (assumed to be regular), and at a sufficiently great distance are independent of the time. The question arises whether more than one system of quantities could satisfy these conditions. The answer is negative.

Since the other quantities are expressible in terms of the potentials, this is most easily proved by showing that the potentials are unique. They satisfy the equations

$$\mathbf{B} = \nabla \times \mathbf{A},$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t},$$

$$0 = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}.$$

If there were two solutions $\mathbf{A}_1$, ϕ_1 and $\mathbf{A}_2$, ϕ_2 for given values of $\mathbf{E}$ and $\mathbf{B}$ the differences

$$\mathbf{A} = \mathbf{A}_2 - \mathbf{A}_1, \qquad \phi = \phi_2 - \phi_1$$

would satisfy the equations

$$\nabla \times \mathbf{A} = 0,$$

$$\nabla\phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = 0,$$

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0.$$

The first equation shows there is a function ψ such that

$$\mathbf{A} = \nabla\psi. \tag{43}$$

The second then becomes

$$\nabla\left(\phi + \frac{1}{c}\frac{\partial\psi}{\partial t}\right) = 0,$$

whence

$$\phi + \frac{1}{c}\frac{\partial\psi}{\partial t} = k,$$

where k depends only on the time. At a sufficiently great distance the left side of this equation is independent of the time. Hence $k = 0$ and

$$\phi = -\frac{1}{c}\frac{\partial\psi}{\partial t}. \tag{44}$$

Substituting this in the third equation we have

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0.$$

This equation is satisfied except possibly on certain regular surfaces where ψ and its first derivatives are continuous. Kirchhoff's formula shows that such a function has the value

$$\psi = \frac{1}{4\pi}\int_S\left[\frac{1}{r}\frac{\partial\psi'}{\partial n} - \left(\psi' + \frac{r}{c}\frac{\partial\psi'}{\partial t'}\right)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]da,$$

where S is a sphere enclosing the region considered. When S is made sufficiently large the time disappears from the integral and this becomes Green's formula for a harmonic function. Since ψ is harmonic everywhere with limit zero at infinity, it is zero. Equations (43) and (44) show that $\mathbf{A}$ and ϕ are then zero, which was to be proved.

A METHOD OF EVALUATING THE FOURIER INTEGRAL[1]

By J. B. Russell[2]

Introduction. In the past few years much interest has been centered on the use of the Fourier integral in problems of electrophysics. In particular, the Fourier integral is a very useful tool in the study and solution of electrical networks.[3] For this reason it is very desirable to have a practical method of evaluating it.

The formal evaluation of the Fourier integral

$$\int_{-\infty}^{\infty} f(\omega)e^{j\omega t}\,d\omega \tag{1}$$

is in general a problem for skilled mathematicians. Even for fairly simple forms of $f(\omega)$ the evaluation of the integral is difficult. The number of known solutions is limited, and practically all of these have been collected and published in an excellent table by Campbell and Foster.[4] By making use of the various properties of the Fourier integral, it is possible to use this table to solve many problems. However, there are still a great many problems which are too complicated to be adapted to the use of the table or are in the form of calculated or experimental curves. For such problems it is necessary to evaluate the Fourier integral by graphical means. This may be done in several ways including the use of calculating machines.[5] It is the purpose of this paper to describe one such method.

[1] This paper is a portion of the research submitted to the Electrical Engineering Department of the Massachusetts Institute of Technology for the degree of Master of Science.

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[3] Representative material discussing the application of the Fourier Integral to electric circuit problems will be found in Berg, Bush I, Campbell and Foster, Carson and Shea. See bibliography for the detailed description of the references.

[4] See bibliography.

[5] See for example; Bauer, Bush II, and Gray.

The General Method. If it is possible to expand $f(\omega)$ in the Fourier integral (1) into a series of functions whose Fourier integrals have been evaluated, then the solution of (1) may be given as a series. For the usual form of $f(\omega)$ found in practical problems, it is very desirable to use a series of orthogonal functions. The first use of such a method is described by Fry.[6] He makes a change of variable such that a series of sines and cosines may be used, the coefficients being evaluated by any suitable means. This method has the advantage that the coefficients of the series may be easily determined, but the disadvantage that the result is given as a series of Bessel functions, thereby increasing the difficulties of numerical computation.

One possible series of orthogonal functions was suggested by Dr. Wiener of the Massachusetts Institute of Technology and is also indicated in the table of Campbell and Foster.[7] These are the functions of Hermite.[8] They have the advantage that when placed in the Fourier integral (1) the resulting function is the same Hermite function as the original, except for a constant factor which depends only on the order of the function. They possess another advantage in that for most practical problems the Hermite series of $f(\omega)$ converges quite rapidly. However, the series of Hermite functions has a serious disadvantage. It is the difficulty of determining the coefficients of the series. Such a determination requires the evaluation of the integral of the product of two functions, which presents practical difficulties of its own. This use of Hermite functions has been developed by the writer to the extent that it is evident that their use is, in general, not practicable.

Another series of orthogonal functions has been derived, however, which are somewhat more useful than the two just considered. They are the following sines and cosines:

$$sin(2n\ tan^{-1}\omega) \tag{2}$$

[6] Fry I.
[7] Page 38, pair 112.
[8] The definition and properties of the Hermite function are given in Courant-Hilbert, pages 77–78.

and

$$cos(2n\ tan^{-1}\omega) \tag{3}$$

where n is any positive integer. These functions are orthogonal with respect to the variable $(2\ tan^{-1}\omega)$ in the interval from $2\ tan^{-1}\omega = -\pi$ to $2\ tan^{-1}\omega = \pi$, that is, from $\omega = -\infty$ to $\omega = \infty$. They possess the advantage that the coefficients of the series may be determined by a harmonic analyser or a schedule harmonic analysis.[9] To facilitate the evaluation of the coefficients by a harmonic analyser an anti-tangent plotting paper has been ruled, thereby simplifying the plotting. When evaluating the coefficients by a schedule analysis, which is usually the more desirable way, only the values of $f(\omega)$ corresponding to the required number of ω's need be known. The functions derived from the Fourier integrals of (2) and (3) lend themselves to numerical calculation easier than those of the first method and practically the same as the Hermite functions. The functions (2) and (3) are also very useful in other ways which can not be taken up in this paper.

The Fourier integral of the sine and cosine of $(2n\ tan^{-1}\omega)$. In order to make use of the functions given by (2) and (3) it is necessary to evaluate their Fourier integrals. The sine function will be considered first. The integral to be evaluated is,

$$\int_{-\infty}^{\infty} sin\ (2n\ tan^{-1}\omega)e^{j\omega t}\,d\omega \tag{4}$$

Defining θ by the relation

$$\theta = 2\ tan^{-1}\omega \tag{5}$$

suggests the derivation of the following expressions:

$$e^{j\theta} = \frac{1 + j\omega}{1 - j\omega} \tag{6}$$

[9] See Lipka, pages 170–208.

and

$$\theta = jln\left[\frac{1 - j\omega}{1 + j\omega}\right] \tag{7}$$

By making use of this expression for θ the integral (4) may be written in a much simpler form. It is,

$$\frac{j}{2}\int_{-\infty}^{\infty}\left(\left[\frac{1 - j\omega}{1 + j\omega}\right]^n - \left[\frac{1 + j\omega}{1 - j\omega}\right]^n\right)e^{j\omega t}\,d\omega \tag{8}$$

A simple change of variables gives the more convenient form,

$$\frac{1}{2}\int_{-j\infty}^{j\infty}\left(\left[\frac{1 - z}{1 + z}\right]^n - \left[\frac{1 + z}{1 - z}\right]^n\right)e^{zt}\,dz \tag{9}$$

The integral (9) may be readily evaluated by making use of the Cauchy integral formula[10] which states that

$$\int_c \frac{f(z)}{(z - a)^n}\,dz = \frac{2\pi j}{(n - 1)!}\frac{d^{n-1}}{dz^{n-1}}\left[f(z)\right]_{z=a} \tag{10}$$

where the integral is taken around a closed path in a region which is analytic[11] except for a finite order pole at $z = a$ enclosed by the path.

In applying this expression to the evaluation of the integral (9) a path of integration is chosen which includes the imaginary axis. Such a path is the one given in Fig. 1, where R is allowed to become infinite. The semicircular part of the path must be taken in the left half of the complex plane, for positive values of t, since the exponential in (9) requires that (zt) must not have a positive real part. A comparison of (9) and (10) shows that

$$f(z) = \frac{(1 - z)^n}{2}e^{zt} - \frac{(1 + z)^{2n}}{2(1 - z)^n}e^{zt} \tag{11}$$

[10] See Hurwitz Courant, page 295.
[11] For the definition of an analytic function see Hurwitz-Courant, page 43.

and that the pole, of nth order, in the integrand of (9) enclosed by the path of integration is at $z = -1$. In other words, $a = -1$. The value of the integral around the path of Fig. 1 is then,

$$\frac{\pi j}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(1-z)^n e^{zt}\right]_{z=-1} \tag{12}$$

The contribution of the second term of (11) is zero, due to the fact that the factor $(1 + z)$ is zero for $z = -1$. Also, the value of this line integral is independent of R as long as it is greater than one, so that the path encloses the pole at $z = -1$.

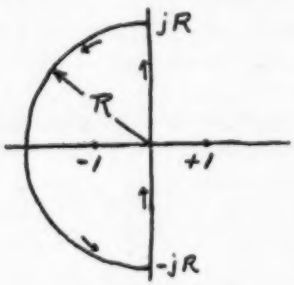

FIG. 1

As R is allowed to become infinite, the integral around the closed path given becomes the sum of the integral (9) and the integral over the semi-circle of infinite radius. It can be shown that the integral over the semi-circle is zero.[12] The value of (9) may therefore be written as,

$$\int_{-\infty}^{\infty} \sin(2n \tan^{-1}\omega)\, e^{j\omega t}\, d\omega = \pi j R_{n-1}\, e^{-t} \tag{13}$$

where R_n is a polynomial defined by,

$$R_n = \frac{e^t}{n!}\frac{d^n}{dz^n}\left[(1-z)^{n+1}\, e^{zt}\right]_{z=-1} \tag{14}$$

[12] A very good discussion of this subject is given in Fry II.

Since ω in (13) may be replaced by $-\omega$, a more general expression may be written. It is,

$$\int_{-\infty}^{\infty} sin(2n\,tan^{-1}\omega)e^{\pm j\omega t}\,d\omega = \pm \pi j R_{n-1}\,e^{-t} \tag{15}$$

where t is positive.

The Fourier integral to be elevated for the cosine term is,

$$\int_{-\infty}^{\infty}\left[(-1)^{n+1} + cos(2n\,tan^{-1}\omega)\right]e^{j\omega t}\,d\omega \tag{16}$$

The constant term $(-1)^{n+1}$ must be included to eliminate the singularity at infinity. It serves to make the integrand zero for $\omega = \pm\infty$. The actual evaluation of (16) can be carried through in the same manner as for (15). The resulting expression is,

$$\int_{-\infty}^{\infty}\left[(-1)^{n+1} + cos(2n\,tan^{-1}\,\omega)\right]e^{\pm j\omega t}\,d\omega = \pi R_{n-1}e^{-t} \tag{17}$$

where t is positive.

The properties of the R_n polynomials. The desired values of the Fourier integrals of (2) and (3) are given by (15) and (16), but they are not in a form which is readily useable. It is desirable to put expression (14) in an entirely different form. The expression may be somewhat simplified by replacing $(1 - z)$ by x. This gives,

$$R_n = \frac{(-1)^n e^{2t}}{n!}\frac{d^n}{dx^n}\left[x^{n+1}\,e^{-xt}\right]_{x=2} \tag{18}$$

By noting that $(-1)^n x^n e^{-xt}$ is the nth derivative of e^{-xt} with respect to t, (18) may be written as,

$$R_n = -\frac{e^{2t}}{n!}\frac{d^{n+1}}{dt^{n+1}}\frac{d^n}{dx^n}\left[e^{-xt}\right]_{x=2} \tag{19}$$

When the differentiation with respect to x is performed there results,[13]

$$R_n = \frac{(-1)^{n+1}e^{2t}}{n!}\frac{d^{n+1}}{dt^{n+1}}\left[t^n e^{-2t}\right] \tag{20}$$

This expression is much easier to handle than the previous one.

It is known that R_n is a polynomial, and its general form may be determined by making use of the expression for the nth derivative of a product. This expression[14] is,

$$\frac{d^n}{dt^n}\left[u \cdot v\right] = v\frac{d^n u}{dt^n} + n\frac{dv}{dt}\frac{d^{n-1}u}{dt^{n-1}} + \frac{n(n-1)}{2!}\frac{d^2 v}{dt^2}\frac{d^{n-2}u}{dt^{n-2}} + \cdots + \frac{d^n v}{dt^n}u \tag{21}$$

Applying this to (20) by letting v be t^n and u be e^{-2t} gives,

$$R_n = \frac{2^{n+1}}{n!}t^n - \frac{2^n}{(n-1)!}C_1^{n+1}t^{n-1} + \frac{2^{n-1}}{(n-2)!}C_2^{n+1}t^{n-2} + \cdots + (-1)^n(n+1)2 \tag{22}$$

$$= \sum_0^n (-1)^{n-\kappa}\frac{2^{\kappa+1}}{K!}C_{\kappa+1}^{n+1}t^{\kappa} \tag{23}$$

where C_m^n is the binomial coefficient. The first few of the R_n polynomials are given below:

$$R_0 = 2$$

$$R_1 = 4t - 4$$

$$R_2 = 4t^2 - 16t + 6$$

[13] The Rn polynomial is related to the Laguerre polynomial, as defined by Courant-Hilbert on pages 79–80. The relation is easily derived from (20) and is,

$$Rn = (-1)^{n+1}\frac{2}{n!}\left[(L'_n(2x) - L_n(2x)\right]$$

[14] Pierce, page 99, No. 858.

$$R_3 = \frac{8}{3}t^3 - 16t^2 + 24t - 8$$

$$R_4 = \frac{4}{3}t^4 - \frac{40}{3}t^3 + 40t^2 - 40t + 10$$

etc.

For certain purposes, particularly that of calculating a table of $R_n e^{-t}$, it is very desirable to have a recurrence formula for these polynomials. This may be obtained by combining various forms of (20). Performing one of the indicated differentiations, considering the product as $(t^n \cdot e^{-2t})$, gives

$$R_n = \frac{(-1)^n}{n!} e^{2t} \frac{d^n}{dt^n}\left[t^n e^{-2t}\right] - \frac{(-1)^n}{(n-1)!} e^{2t} \frac{d^n}{dt^n}\left[t^{n-1} e^{-2t}\right] \quad (24)$$

If this expression is divided by e^{2t} and then differentiated with respect to t the resulting terms are recognizable, and give,

$$\frac{d}{dt}\left[R_n e^{-2t}\right] = -2\left[R_n e^{-2t}\right] - \frac{d}{dt}\left[R_{n-1} e^{-2t}\right] \quad (25)$$

A similar differentiation of the product in (20) may be carried out, considering the product to be $(t \cdot t^{n-1} e^{-2t})$. The result is, after dividing the expression by e^{2t},

$$\left[R_n e^{-2t}\right] = \frac{-t}{n} \frac{d}{dt}\left[R_{n-1} e^{-2t}\right] - \frac{n+1}{n}\left[R_{n-1} e^{-2t}\right] \quad (26)$$

Combining (25) and (26), to eliminate the derivative terms, gives the desired recurrence formula. It is,

$$nR_n + 2(n-t) R_{n-1} + nR_{n-2} = 0 \quad (27)$$

The differential equation of the R_n polynomial may be derived by a similar process, and is,

$$t \frac{d^2}{dt^2} R_n - 2(t-1) \frac{d}{dt} R_n + 2nR_n = 0 \quad (28)$$

Summary of Method. The general problem is to evaluate the Fourier integral

$$\int_{-\infty}^{\infty} f(\omega)\, e^{j\omega t}\, d\omega \tag{29}$$

where $f(\omega)$ is considered to be a real function. If it is complex its real and imaginary parts may be handled separately.

To apply the method just developed the function $f(\omega)$ is expressed as a function of the angle θ, defined by

$$\theta = 2\ tan^{-1}\ \omega \tag{30}$$

This function is then expanded[15] into a sine and cosine series of the angle θ, by any convenient method of harmonic analysis. Let this series be,

$$f(\omega) = F(\theta) = \sum_{1}^{\infty} a_n\ sin\ n\theta + \sum_{1}^{\infty} b_n\ [(-1)^{n+1} + cos\ n\theta] \tag{31}$$

The (-1) term is included in the cosine part of the series in order that it will be zero at $\omega = \pm\infty$, as is required by the form of (17). Such a restriction is fulfilled by most of the $f(\omega)$'s occurring in practical problems. If this restriction is not satisfied $f(\omega)$ may be expressed as the sum of three functions; the first, a function which is zero at $\omega = \pm\infty$ and therefore satisfies the restriction; the second, a step function at $\omega = 0$ of magnitude $f(+\infty) - f(-\infty)$; and the third, a constant of $\frac{f(+\infty) + f(-\infty)}{2}$. Only the first[16] of these functions will be considered since the Fourier integrals[17] of the other two are known.

[15] It is tacitly assumed that such an expansion is possible. For a discussion of the existance and convergence of Fourier series see, Byerly, Carslaw, Hobson or Wiener. In any practical problem the necessary properties of convergence are obvious.

[16] For a discussion of the other two functions as applied to electric circuit problems see Russell, pages 19–26.

[17] These Fourier integrals are given in Campbell-Foster; for the constant, page 43, pair 403.1; for the step function, page 44, pair 415.

By making use of (15) and (17) the value of the Fourier integral (29) may be expressed in terms of R_n and the Fourier coefficients of (31). This expression is,

$$\int_{-\infty}^{\infty} f(\omega)\, e^{-j\omega t}\, d\omega = \pi \sum_{1}^{\infty} b_n R_{n-1} e^{-t} + j\pi \sum_{1}^{\infty} a_n R_{n-1} e^{-t} \quad (32)$$

for positive values of t, and

$$\int_{-\infty}^{\infty} f(\omega)\, e^{-j\omega t}\, d\omega = \pi \sum_{1}^{\infty} b_n R_{n-1} e^{-t} - j\pi \sum_{1}^{\infty} a_n R_{n-1} e^{-t} \quad (33)$$

for positive values of t. These expressions lend themselves quite readily to numerical computations as long as the terms converge rapidly enough. It can be shown by a consideration of the maximum values of the functions $R_n e^{-t}$ that the series of (32) and (33) will converge at least as rapidly as the series $\Sigma n a_n$ and $\Sigma n b_n$. In other words, if the nth Fourier coefficient of (32) is 1% of the first, then the maximum value of the nth term in the corresponding R_n series will be less than n% of the maximum value of the first term. This fact is useful in determining the number of terms which are necessary for the desired accuracy.

In many problems the series involved can be made to converge more rapidly by changing the scale of ω. If, as has been previously assumed, $f(\omega)$ is expressed as a function of $(2\ tan^{-1}\omega)$ a particular set of values will be found for the Fourier coefficients. However, if $f(\omega)$ is expressed as a function of $(2\ tan^{-1}\kappa\omega)$ it is usually possible to choose a value of κ which will give nearly the most convergent series. This means that the same Fourier integral may be evaluated as any number of different R_n series, the differences between these series being the manner in which they converge and the scale of the resulting functions. It is, of course, desirable to choose the value of κ which gives the most convergent series. This value can be chosen accurately enough from the general shape of the function $f(\omega)$.

When such a change in the scale of ω is made the integral actually evaluated by the method described is,

$$H(t_1) = \int_{-\infty}^{\infty} F(\omega_1)\, e^{j\omega t_1}\, d\omega_1 \tag{34}$$

where

$$\omega_1 = \kappa\omega$$

and

$$F(\omega_1) = f\left(\frac{\omega_1}{\kappa}\right)$$

The integral whose value is desired is,

$$\mathrm{h}(t) = \int_{-\infty}^{\infty} f(\omega)\, e^{j\omega t}\, d\omega \tag{35}$$

By replacing ω_1 by $\kappa\omega$ and t_1 by $\frac{t}{\kappa}$ in (34) it can be seen by comparison with (35) that

$$h(t) = \frac{1}{\kappa} H\left(\frac{t}{\kappa}\right) \tag{36}$$

This means that when (29) is evaluated by the Fourier coefficients derived from $f(\omega)$ expressed as a function of $(2t\tan^{-1}\kappa\omega)$ the correct result is given by (32) or (33) if they are multiplied by $\frac{1}{\kappa}$ and t is replaced by $\frac{t}{\kappa}$.

Illustrative Example. It is interesting to apply this method to the solution of a simple electric circuit problem. Consider the circuit given in Fig. 2. It is desired to find the indicial admittance of this circuit for the two values of resistance, that is, the current flowing as a result of a suddenly applied steady voltage of one volt.

The indicial admittance of an electric circuit is given by the following Fourier integral,[18]

$$A(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{Q(\omega)}{\omega} e^{j\omega t}\, d\omega \tag{37}$$

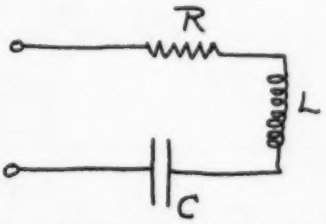

FIG. 2. $L = 1$h. $C = 1\mu$f. $R = 2000\,\omega$ or $4000\,\omega$

where $Q(\omega)$ is the imaginary part of the usual complex, or *A.C.*, admittance function. For the problem given this admittance function is,

$$f(\omega) = \frac{j\omega}{(j\omega)^2 L + j\omega \frac{R}{L} + \frac{1}{LC}} \tag{38}$$

$$= \frac{j\omega}{(j\omega)^2 + j\omega R + 10^6} \tag{39}$$

Taking the real part of (39) and substituting in (37) gives,

$$A(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{10^6 - \omega^2}{(10^6 - \omega^2)^2 + \omega^2 R^2} e^{j\omega t}\, d\omega \tag{40}$$

This is the Fourier integral to be evaluated.[19]

A brief consideration of the function, $\frac{Q(\omega)}{\omega}$ indicates that a desirable series expansion is in terms of the angle $(2 \tan^{-1} \omega\, 10^{-3})$.

[18] See Bush I, pages 178–180, or Russell, pages 21–25.

[19] The value of this integral may also be found in Campbell-Foster, page 46, pairs 448 and 449.

This means that κ, the scale factor for ω, has been chosen as 10^{-3}. For the sake of interest $\frac{Q(\omega)}{\omega}$ has been plotted to this scale, for both values of R, in Fig. 3. The Fourier coefficients were evaluated by a harmonic schedule using twenty-four ordinates, thus giving twelve coefficients. The required values of $\frac{Q(\omega)}{\omega}$ were

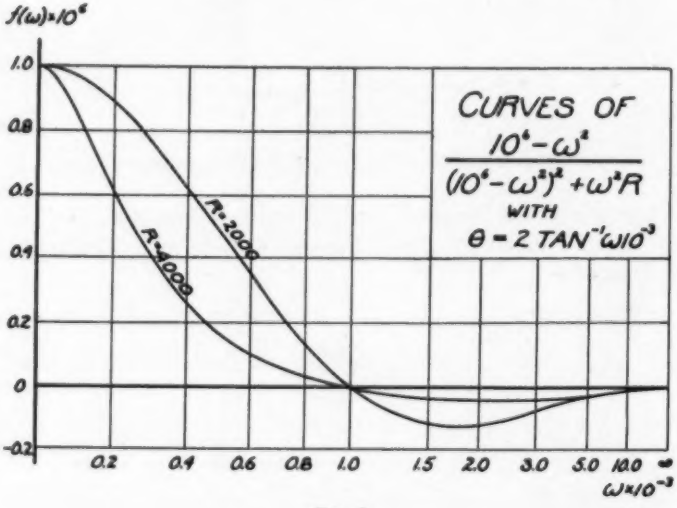

FIG. 3

at points equally spaced with respect to the angle $(2 \tan^{-1} \omega\, 10^{-3})$ and were, therefore, at the following values of ω.

$$\omega = 10^3 \tan \frac{n\pi}{24}$$

where

$$n = -12, -11, \ldots, 0, \ldots . 11, 12$$

The computed numerical values of the coefficients[20] are,

For $R = 2000\,\omega$	For $R = 4000\,\omega$
$b_1 = 5.00 \times 10^{-7}$	$b_1 = 3.333 \times 10^{-7}$
$b_2 = 2.50 \times 10^{-7}$	$b_2 = 2.222 \times 10^{-7}$
$b_3 = b_4 = \ldots = 0$	$b_3 = 1.111 \times 10^{-7}$
$a_1 = a_2 = \ldots = 0$	$b_4 = 7.406 \times 10^{-8}$
	$b_5 = 3.703 \times 10^{-8}$
	$b_6 = 2.471 \times 10^{-8}$
	$b_7 = 1.249 \times 10^{-8}$
	$b_8 = 8.33 \times 10^{-9}$
	$b_9 = 4.29 \times 10^{-9}$
	$b_{10} = 3.08 \times 10^{-9}$
	$b_{11} = 1.86 \times 10^{-9}$
	$b_{12} = 9.3 \times 10^{-10}$
	$a_1 = a_2 = \ldots = 0$

By means of (32) and (36) the desired indicial admittances may be written down, and are

For $R = 2000\,\omega$

$$A(t) = 10^3 \sum_{1}^{\infty} b_n R_{n-1}(t\,10^3)\, e^{-10^3 t}$$

$$= te^{-10^3 t} \tag{41}$$

[20] It is interesting to note that the sine terms are zero. This would be expected since $A(t)$ must be real. Also, if the real part of $\frac{f(\omega)}{\omega}$ is expressed as a Fourier series of $(2 \tan^{-1} \omega\, 10^{-3})$ only sine terms result and the coefficients are the same as the corresponding ones in the cosine series of $\frac{Q(\omega)}{\omega}$.

For $R = 4000\ \omega$

$$A(t) = 10^3 \sum_{1}^{\infty} b_n R_{n-1} (t\,10^3)\, e^{-10^3 t} \tag{42}$$

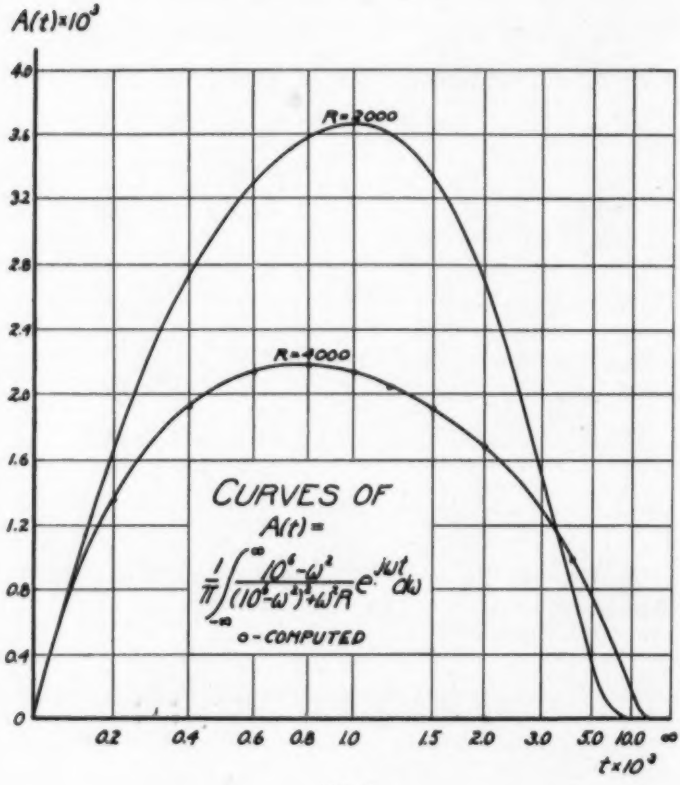

FIG. 4

It will be seen that (41) is exactly the correct expression. The actual numerical value of (42) has been computed for a number of points. These are plotted, with the correct values for comparison, in Fig. 4. The numerical value of (41) has also been

plotted. The anti-tangent scale for t has been used simply for convenience.

Conclusions. The method of evaluating a Fourier integral, which has been described in this paper, is applicable to a variety of problems. The accuracy of the result is, in general, dependent only on the number of terms in the series which it is practical to use. In cases where the series does not converge rapidly enough it is usually possible to overcome the difficulty by making use of known Fourier integrals to evaluate part of the desired integral. The coefficients of each term can be computed by the ordinary methods of harmonic analysis with very little work. The numerical computation of the result is similar to that of evaluating any series, but will be greatly simplified when a table of $R_n e^{-t}$ is available.

This method should extend the range of problems which can be handled by means of the Fourier integral.

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A TABLE OF HERMITE FUNCTIONS[1]

By J. B. Russell[2]

The Hermite functions, whose numerical values are given in this table, are defined by the relation,[3]

$$\phi_n(\chi) = e^{-\frac{\chi^2}{2}} H_n(\chi)$$

where $H_n(\chi)$ is the Hermite polynomial defined by,

$$H_n(\chi) = \frac{d^n}{dt^n}\left(e^{-t^2+2t\chi}\right)_{t=0}$$

The general form of the polynomial is given by,

$$H_n(\chi) = 2^n\chi^n - 2^{n-1}C_2^n\chi^{n-2} + 1\cdot 3\cdot 2^{n-2}C_4^n\chi^{n-4} - \ldots$$

where C_m^n is the binomial coefficient.

Note. The small raised figure which appears in the absence of a decimal point gives the number of zeros between the decimal point and the significant figures given.

[1] This table was computed as part of a research thesis presented to the Electrical Engineering Department of the Massachusetts Institute of Technology for the degree of Master of Science.

[2] Instructor in Electrical Engineering, Columbia University.

[3] Courant-Hilbert, Methoden der Mathematischen Physik, Julius Springer, 1931, pages 77–78.

x	$\phi_0(x)$	$\phi_1(x)$	$\phi_2(x)$	$\phi_3(x)$
0.00	1.0000	0.0000	−2.0000	0.0000
0.04	.99920	^{1}79936	−1.9920	−.47910
0.08	.99681	.15949	−1.9681	−.95285
0.12	.99283	.23828	−1.9285	−1.4159
0.16	.98728	.31593	−1.8735	−1.8632
0.20	.98020	.39208	−1.8036	−2.2897
0.24	.97161	.46637	−1.7194	−2.6908
0.28	.96156	.53847	−1.6216	−3.0620
0.32	.95009	.60806	−1.5110	−3.3993
0.36	.93726	.67482	−1.3886	−3.6991
0.40	.92312	.73849	−1.2554	−3.9583
0.44	.90774	.79881	−1.1125	−4.1743
0.48	.89119	.85554	−.96106	−4.3448
0.53	.87354	.90848	−.80226	−4.4683
0.56	.85488	.95746	−.63740	−4.5437
0.60	.83527	1.0023	−.46775	−4.5706
0.64	.81481	1.0430	−.29464	−4.5490
0.68	.79358	1.0793	−.11936	−4.4794
0.72	.77167	1.1112	+^{1}56795	−4.3630
0.76	.74916	1.1387	.23254	−4.2014
0.80	.72615	1.1618	.40664	−3.9967
0.84	.70272	1.1806	.57792	−3.7514
0.88	.67896	1.1950	.74522	−3.4683
0.92	.65495	1.2051	.90750	−3.1506
0.96	.63078	1.2111	1.0638	−2.8020
1.00	.60653	1.2131	1.2131	−2.4261
1.10	.54608	1.2014	1.5509	−1.3936
1.20	.48675	1.1682	1.8302	−.20837
1.30	.42956	1.1169	2.0447	+.84881
1.40	.37531	1.0509	2.1918	1.9336
1.50	.32465	.97396	2.2726	2.9219
1.60	.27804	.88972	2.2910	3.7724
1.70	.23575	.80154	2.2537	4.4565
1.80	.19790	.71244	2.1690	4.9586
1.90	.16448	.62501	2.0461	5.2750
2.00	.13534	.54134	1.8947	5.4134
2.10	.11025	.46305	1.7243	5.3900
2.20	^{1}88922	.39126	1.5437	5.2272
2.30	^{1}71005	.32663	1.3605	4.9516
2.40	^{1}56135	.26945	1.1811	4.5914

x	$\phi_0(x)$	$\phi_1(x)$	$\phi_2(x)$	$\phi_3(x)$
2.50	$^{1}43927$	.21964	1.0103	4.1731
2.60	$^{1}34048$	.17705	.85255	3.7251
2.70	$^{1}26121$	.14106	.70946	3.2669
2.80	$^{1}19841$	.11111	.58254	2.8178
2.90	$^{1}14921$	$^{1}86541$	.47209	2.3920
3.00	$^{1}11109$	$^{1}66654$	.37771	1.9996
3.10	$^{2}81887$	$^{1}50770$	.29840	1.6470
3.20	$^{2}59760$	$^{1}38247$	.23283	1.3371
3.30	$^{2}43178$	$^{1}28498$	.17945	1.0704
3.40	$^{2}30887$	$^{1}21003$	.13665	.84517
3.50	$^{2}21875$	$^{1}15312$	.10281	.65843
3.60	$^{2}15338$	$^{1}11043$	$^{1}76445$	.50623
3.70	$^{2}10648$	$^{2}78793$	$^{1}56177$	.38420
3.80	$^{2}73180$	$^{2}55617$	$^{1}40805$	.28787
3.90	$^{2}49796$	$^{2}38841$	$^{1}29300$	.21300
4.00	$^{3}33546$	$^{2}26837$	$^{1}20799$	.15566
4.20	$^{3}14775$	$^{2}12411$	$^{1}10130$	$^{1}80124$
4.40	$^{4}62522$	$^{3}55019$	$^{2}47166$	$^{1}39493$
4.60	$^{4}25419$	$^{3}23386$	$^{2}21007$	$^{1}18391$
4.80	$^{5}99295$	$^{4}95323$	$^{3}89524$	$^{2}82131$
5.00	$^{5}37267$	$^{4}37267$	$^{3}36521$	$^{3}35031$
5.20	$^{5}13438$	$^{4}13976$	$^{3}14266$	$^{3}14277$
5.40	$^{6}46557$	$^{5}50282$	$^{4}53373$	$^{3}55632$
5.60	$^{6}15498$	$^{5}17357$	$^{4}19130$	$^{3}20732$
5.80	$^{7}49564$	$^{6}57494$	$^{5}65702$	$^{4}73915$
6.00	$^{7}15230$	$^{6}18276$	$^{5}21627$	$^{4}25221$
6.20	$^{8}44964$	$^{7}55755$	$^{6}68237$	$^{5}82383$
6.40	$^{8}12754$	$^{7}16325$	$^{6}20641$	$^{5}25768$
6.60	$^{9}31452$	$^{8}41516$	$^{7}54172$	$^{6}69850$
6.80	$^{10}91015$	$^{8}12378$	$^{7}16652$	$^{6}22152$
7.00	$^{10}22898$	$^{9}32057$	$^{8}44422$	$^{7}60908$
7.50	$^{12}61019$	$^{11}91529$	$^{9}13607$	$^{8}20045$
8.00	$^{13}12664$	$^{12}20263$	$^{11}32167$	$^{10}50657$

x	$\phi_4(x)$	$\phi_5(x)$	$\phi_6(x)$	$\phi_7(x)$
0.00	12.000	0.0000	−120.00	0.0000
0.04	11.913	4.7859	−118.75	−66.931
0.08	11.656	9.4878	−115.04	−132.22
0.12	11.231	14.023	−108.94	−194.42
0.16	10.645	18.312	−100.59	−251.93
0.20	9.9055	22.281	−90.143	−303.45
0.24	9.0246	25.858	−77.834	−347.66
0.28	8.0148	28.984	−63.916	−383.60
0.32	6.8906	31.604	−48.679	−410.41
0.36	5.6685	33.674	−32.439	−427.45
0.40	4.3660	35.159	−15.532	−434.34
0.44	3.0018	36.036	+1.6934	−430.94
0.48	1.5954	36.290	19.235	−417.35
0.52	+.16655	35.920	35.691	−393.92
0.56	−1.2646	34.933	51.772	−361.22
0.60	−2.6782	33.351	66.803	−320.05
0.64	−4.0549	31.201	80.486	−271.40
0.68	−5.3759	28.524	92.551	−216.42
0.72	−6.6235	25.366	102.76	−156.42
0.76	−7.7815	21.784	110.93	−92.799
0.80	−8.8346	17.838	116.89	−27.041
0.84	−9.7698	13.598	120.54	+39.338
0.88	−10.576	9.1332	121.83	104.82
0.92	−11.242	+4.5195	120.74	167.94
0.96	−11.762	−.16768	117.30	227.23
1.00	−12.131	−4.8523	111.60	281.43
1.10	−12.371	−16.068	88.362	387.21
1.20	−11.654	−25.727	54.796	440.23
1.30	−10.061	−32.950	+14.943	434.24
1.40	−7.7368	−37.132	−26.601	371.10
1.50	−4.8698	−37.988	−65.255	260.05
1.60	−1.6745	−35.538	−96.975	+116.13
1.70	+1.6299	−30.111	−118.68	−42.166
1.80	4.8370	−22.255	−128.49	−195.50
1.90	7.7688	−12.679	−125.87	−326.15
2.00	10.286	−2.1654	−111.52	−420.08
2.10	12.292	+8.5061	−87.193	−468.28
2.20	13.738	18.628	−55.414	−467.35
2.30	14.615	27.615	−19.120	−419.33
2.40	14.952	35.056	+18.748	−326.14

x	$\phi_4(x)$	$\phi_5(x)$	$\phi_6(x)$	$\phi_7(x)$
2.50	14.803	40.632	55.128	−211.95
2.60	14.255	44.326	87.943	−74.604
2.70	13.684	46.140	115.31	+69.015
2.80	12.284	46.250	136.16	207.48
2.90	11.041	44.902	150.02	331.30
3.00	9.7315	42.392	157.04	433.52
3.10	8.4209	39.034	157.80	509.96
3.20	7.1605	35.129	153.22	559.06
3.30	5.9877	30.956	144.43	581.79
3.40	4.9273	26.744	132.59	580.67
3.50	3.9922	22.678	118.82	559.62
3.60	3.1862	18.891	104.15	523.20
3.70	2.5060	15.471	89.398	475.95
3.80	1.9430	12.464	75.295	422.68
3.90	1.4856	9.8838	62.237	366.65
4.00	1.1205	7.7183	50.542	311.72
4.20	.61227	4.5020	31.695	212.21
4.40	.31924	2.4934	18.749	135.07
4.60	.15659	1.2935	10.334	79.553
4.80	$^{1}73474$	.63964	5.4059	44.220
5.00	$^{1}32839$	.30037	2.6753	23.149
5.20	$^{1}13993$	.13399	1.2536	11.430
5.40	$^{2}56779$	$^{1}56871$	.55743	5.3378
5.60	$^{2}22071$	$^{1}23061$	.23621	2.3688
5.80	$^{2}81799$	$^{2}88973$	$^{1}95030$	.99557
6.00	$^{2}28968$	$^{2}32743$	$^{1}36395$	.39745
6.20	$^{4}98061$	$^{2}11501$	$^{1}13280$	.15087
6.40	$^{4}31744$	$^{3}38571$	$^{2}46197$	$^{1}54503$
6.60	$^{4}88946$	$^{3}11182$	$^{2}13871$	$^{1}16968$
6.80	$^{5}29127$	$^{4}37845$	$^{2}48559$	$^{2}61498$
7.00	$^{6}82607$	$^{4}11078$	$^{3}14683$	$^{2}19226$
7.50	$^{7}29251$	$^{6}41663$	$^{5}59569$	$^{4}84749$
8.00	$^{9}79129$	$^{7}12254$	$^{6}18815$	$^{5}28634$

x	$\phi_8(x)$	$\phi_9(x)$	$\phi_{10}(x)$	$\phi_{11}(x)$
0.00	1680.0	0.0000	−30240.	0.0000
0.04	1657.2	1203.5	−29733.	−26448.
0.08	1589.4	2370.5	−28231.	−51926.
0.12	1487.5	3465.6	−25782.	−75500.
0.16	1327.6	4455.8	−22471.	−96306.
0.20	1140.6	5311.5	−18407.	−113591
0.24	922.80	6005.5	−13728.	−126701
0.28	680.01	6518.4	−8589.9	−135181
0.32	418.85	6834.6	−3165.2	−138731
0.36	+146.39	6944.5	+2365.1	−137191
0.40	−130.02	6845.4	7816.7	−130651
0.44	−402.93	6540.4	13008.	−119361
0.48	−665.04	6039.1	17768.	−103731
0.52	−909.34	5356.9	21974.	−84285.
0.56	−1129.4	4514.6	25385.	−61861.
0.60	−1319.3	3537.6	27993.	−37161.
0.64	−1474.2	2455.4	29678.	−11119.
0.68	−1590.1	1300.2	30389.	+15325.
0.72	−1663.2	+107.75	30092.	41177.
0.76	−1694.0	−1090.1	28835.	65632.
0.80	−1679.7	−2254.9	26627.	87700.
0.84	−1621.5	−3353.5	23553.	106641
0.88	−1521.1	−4354.3	19717.	121791
0.92	−1381.3	−5228.7	15243.	132621
0.96	−1205.9	−5951.1	10281.	138761
1.00	−999.56	−6502.0	+4988.1	140021
1.10	−385.21	−7042.7	−8560.3	122021
1.20	+293.79	−6338.6	−20501.	77570.
1.30	919.84	−4556.4	−28484.	+17278.
1.40	1411.5	−1985.4	−30964.	−46997.
1.50	1693.7	+920.39	−27726.	−101591
1.60	1729.3	3675.6	−19365.	−135481
1.70	1518.1	5836.2	−7482.7	−142161
1.80	1095.1	7071.2	+5741.5	−120731
1.90	+522.79	7205.0	17969.	−75983.
2.00	−120.18	6240.6	27121.	−16329.
2.10	−746.09	4359.0	31737.	+46117.
2.20	−1280.6	+1843.2	31160.	100241
2.30	−1661.2	−932.42	25610.	136461
2.40	−1849.8	−3587.9	16074.	148911

x	$\phi_8(x)$	$\phi_9(x)$	$\phi_{10}(x)$	$\phi_{11}(x)$
2.50	−1831.5	−5766.5	+4135.1	13601[1]
2.60	−1619.2	−7225.9	−8430.1	10068[1]
2.70	−1241.7	−7809.5	−19820.	+49160.
2.80	−744.31	−7487.8	−28534.	−10034.
2.90	−178.76	−6337.6	−33540.	−67782.
3.00	+402.59	−4520.7	−34370.	−11581[1]
3.10	952.53	−2253.6	−31118.	−14786[1]
3.20	1432.9	+225.66	−24348.	−16034[1]
3.30	1817.7	2688.4	−14976.	−15261[1]
3.40	2092.3	4937.0	−4059.0	−12655[1]
3.50	2253.9	6823.0	+7191.6	−86119.
3.60	2308.9	8253.1	17861.	−36438.
3.70	2270.5	9186.2	27110.	+16887.
3.80	2158.2	9639.6	34413.	68747.
3.90	1990.1	9653.1	39472.	11482[1]
4.00	1786.2	9301.7	42263.	15207[1]
4.20	1338.8	7850.9	41848.	19451[1]
4.40	926.15	5989.0	36032.	19730[1]
4.60	587.19	4129.3	27420.	16968[1]
4.80	348.83	2641.3	19077.	13032[1]
5.00	194.03	1568.3	12190.	90535.
5.20	101.32	870.83	7232.9	57806.
5.40	49.844	452.91	3994.2	34080.
5.60	23.224	222.21	2070.7	18748.
5.80	10.218	102.60	1006.3	9620.6
6.00	4.2599	44.759	460.43	4630.0
6.20	1.6849	18.479	198.81	2095.6
6.40	.63292	7.2299	81.150	894.12
6.60	.20456	2.4287	28.376	325.99
6.80	[1]76839	.94662	11.491	137.34
7.00	[1]24861	.31730	3.9947	49.579
7.50	[3]11878	[1]16462	.22554	3.0539
8.00	[4]43180	[3]64507	[2]95439	.13980

A THEORY OF GRAVITATION AND ELECTROMAGNETISM

By Hsin P. Soh

1. Einstein's successful geometrization of the gravitational field led at once to the hope that similar purely geometrical ideas might also serve completely to describe the phenomena of the electromagnetic field as embodied in the classical Maxwell-Lorentz equations and Lorentz's law of force on a charged particle. This idea led Weyl and Eddington, soon after the discovery of the geometrical significance of the gravitational field, and more recently Einstein himself, to suggest modifications of Riemann's geometry so as to make room for the electromagnetic field. None of these theories has proved quite satisfactory.[1]

The physical agent inseparably associated with gravitation is *mass*, just as the physical entity inseparably associated with electromagnetism is electric *charge*. If a unified theory of gravitation and electromagnetism is desired it becomes necessary to examine these two fundamental entities in an endeavour to discover their analogies and differences, so as to obtain the best clues for a method of attack.[2] Now the most obvious resemblance is found in the inverse square law of interaction between two masses or two charges. The difference in the manner of their interaction lies in the fact that while two masses necessarily of the same sign attract, two charges of the same sign repel. Thus two charges of the same sign interact in the oppo-

[1] In none of these theories in the connection between the geometrical quantities interpreted as the electromagnetic potential and the charge and current densities made quite clear. In other words, the relation between these quantities and the classical retarded electromagnetic potential is not in the least apparent. Further, no purely geometrical derivation of the Lorentz law of force seems possible in any of these theories.

[2] The simplicity and directness of the results obtained by this method convince me that the theory here presented is probably more than just another purely formal device for obtaining Maxwell's equations.

site way as two masses of the same sign. The difference in the sign of the interaction, and the double sign of the electric charge, can be formally brought out if we associate the factor $\pm \sqrt{-1}$ with the electric charge.

With this idea in mind the present author has been led to study a possible modification of the theory of relativity characterized by a complex Riemannian line element in which the real part is associated with mass (gravitation) and the imaginary part with charge (electromagnetism). The theory thus obtained, to be presented in this paper, is distinguished by a remarkable simplicity; as will be shown the Maxwell equations and the Lorentz law of force both come out in the first approximation as a result of a fundamental identity of Riemannian geometry. Further the connection between charge and current densities, and electromagnetic potential, becomes at once clear.

One is led to believe that the present theory will find its place in a more comprehensive theory of matter, gravitation and electricity, as already forecast by recent developments in quantum theory, by the following argument: In the sense of the correspondence principle one must expect a unified theory of electromagnetism and gravitation for large scale phenomena which will be obtained from the complete (as yet non-existent) theory in the limit $h \to 0$. Now if in this theory Weyl's principle of gauge invariance is right, the gauge exponent must be an imaginary quantity. This necessitates the introduction of a complex line element as done in this paper.[3]

2. We postulate for the physical world a complex four-dimensional Riemannian geometry where, for evident physical reasons, the coordinates are real quantities. A fundamental identity of this geometry is[4] postulated to be

$$G_\nu^\mu - \tfrac{1}{2} g_\nu^\mu G = T_\nu^\mu - \tfrac{3}{4} g_\nu^\mu T \tag{1}$$

[3] I owe this remark to Professor Eddington.

[4] The notation is that used in Eddington's "Mathematical Theory of Relativity."

where

$$T^{\mu\nu} = (\rho + i\sigma)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \qquad i = \sqrt{-1}$$

and ρ and σ are real quantities to be interpreted as the mass and charge density, respectively. The dimensional factors required by the choice of units are omitted.

To find $g_{\mu\nu}$ we assume that $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ where $\delta_{\mu\nu}$ have the Minkowski values and $h_{\mu\nu}$ are small quantities of the first order. Then

$$(2) \qquad G_{\mu\nu} = \tfrac{1}{2}\, g^{\sigma\rho}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho} + \frac{\partial^2 g_{\sigma\rho}}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} - \frac{\partial^2 g_{\nu\rho}}{\partial x^\mu \partial x^\sigma}\right)$$

which to the first approximation can be satisfied by breaking it up into two equations[5]

$$(3) \qquad G_{\mu\nu} = \tfrac{1}{2}\, g^{\sigma\rho}\frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho}$$

$$(4) \qquad 0 = g^{\sigma\rho}\left(\frac{\partial^2 g_{\sigma\rho}}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} - \frac{\partial^2 g_{\nu\rho}}{\partial x^\mu \partial x^\sigma}\right)$$

Both are satisfied if

$$(5) \qquad \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) h_{\mu\nu} = 2\, G_{\mu\nu}$$

It should be noted that the present analysis, as given in Eddington's treatise, l.c., is independent of whether the $h_{\mu\nu}$ are complex or real.

From (1) we obtain

$$(6) \qquad G^{\mu\nu} = T^{\mu\nu} + g^{\mu\nu}\, T$$

which may be written since the $\left(\frac{dx^\nu}{ds}\right)^2 << 1$ and $\frac{ds}{dt} \to 1$

[5] See Eddington's, l.c., p. 128.

$$(7)\quad G^{\mu\nu} = \begin{Vmatrix} 2(\rho + i\sigma) & (\rho + i\sigma)\frac{dx}{dt} & (\rho + i\sigma)\frac{dy}{dt} & (\rho + i\sigma)\frac{dz}{dt} \\ (\rho + i\sigma)\frac{dx}{dt} & -(\rho + i\sigma) & 0 & 0 \\ (\rho + i\sigma)\frac{dy}{dt} & 0 & -(\rho + i\sigma) & 0 \\ (\rho + i\sigma)\frac{dz}{dt} & 0 & 0 & -(\rho + i\sigma) \end{Vmatrix}$$

and

$$(8)\quad G_{\mu\nu} = \begin{Vmatrix} 2(\rho + i\sigma) & -(\rho + i\sigma)\frac{dx}{dt} & -(\rho + i\sigma)\frac{dy}{dt} & -(\rho + i\sigma)\frac{dz}{dt} \\ -(\rho + i\sigma)\frac{dx}{dt} & -(\rho + i\sigma) & 0 & 0 \\ -(\rho + i\sigma)\frac{dy}{dt} & 0 & -(\rho + i\sigma) & 0 \\ -(\rho + i\sigma)\frac{dz}{dt} & 0 & 0 & -(\rho + i\sigma) \end{Vmatrix}$$

The differential equations (5) can be readily solved because they are quite similar to the classical equations for the retarded potentials. We obtain

$$(9)\quad g_{\mu\nu} = \begin{Vmatrix} 1 - 2\mu - 2i\phi & a_x + iA_x & a_y + iA_y & a_z + iA_z \\ a_x + iA_x & -(1 - \mu - i\phi) & 0 & 0 \\ a_y + iA_y & 0 & -(1 - \mu - i\phi) & 0 \\ a_z + iA_z & 0 & 0 & -(1 - \mu - i\phi) \end{Vmatrix}$$

where

$$(10)\qquad \mu \equiv \iiint \left\{ \frac{\rho dv}{r} \right\}_{t-\frac{r}{c}} \qquad a_x \equiv \iiint \left\{ \rho \frac{\frac{dx}{dt}}{r} \right\} dV \quad \text{etc.}$$

$$(11)\qquad \phi \equiv \iiint \left\{ \frac{\sigma dv}{r} \right\}_{t-\frac{r}{c}} \qquad A_x \equiv \iiint \left\{ \sigma \frac{\frac{dx}{dt}}{r} \right\} dV \quad \text{etc.}$$

and constant factors have been omitted by proper choice of the units. Thus to the desired approximation the Riemannian line element is

$$(12)\qquad ds^2 = \{(1-2u)dt^2 + a_x dx dt + a_y dy dt + a_z dz dt - (1-u)(dx^2+dy^2+dz^2)\}$$
$$+ i\{(-2\phi)dt^2 + A_x dx dt + A_y dy dt + A_z dz dt + \phi(dx^2+dy^2+dz^2)\}$$

Thus in the first approximation the gravitational and the electromagnetic fields are kept separate in the Riemannian line element, which is in agreement with experimental facts. The former occurs in the real part of the metric which contains the mass density through the potentials u, a_x, a_y, a_z, and the latter in the imaginary part which contains the charge density through the potentials ϕ, A_x, A_y, A_z. It is seen that the expression for ds^2 does not reduce to the Minkowskian line element unless both the charge and the mass density vanish. However, at any space-time point the real part of the metric may reduce to the Minkowskian ds^2 without thereby destroying the electromagnetic field.

A fundamental identity of Riemannian geometry is that the divergence of the tensor $G^{\mu\nu} - \frac{1}{2} g^{\mu\nu}$ vanishes. From this we obtain[6]

$$(13)\qquad \nabla_\mu (T^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} T) = 0$$

[6] ∇_μ is the symbol for covariant differentiation.

which gives, in the special case wherein ρ and σ are assumed constant:

$$(14) \qquad (\rho + i\sigma)\frac{d^2x}{ds^2} = -(\rho + i\sigma)\left\{ \begin{matrix} \alpha\beta \\ \nu \end{matrix} \right\} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$$

The real part of this expression is, as we shall see, the Lorentz law of force. The physical interpretation of the imaginary part is not at present apparent. To evaluate (14) we first calculate the Christoffel symbols and obtain.

$$\begin{bmatrix} 11 \\ 1 \end{bmatrix} = -\frac{\partial u}{\partial t} - i\frac{\partial \phi}{\partial t} \quad \begin{bmatrix} 11 \\ 2 \end{bmatrix} = \frac{\partial a_x}{\partial t} + \frac{\partial u}{\partial x} + i\left(\frac{\partial A_x}{\partial t} + \frac{\partial \phi}{\partial x}\right)$$

$$(15) \quad \begin{bmatrix} 11 \\ 3 \end{bmatrix} = \frac{\partial a_y}{\partial t} + \frac{\partial u}{\partial y} + i\left(\frac{\partial A_y}{\partial t} + \frac{\partial \phi}{\partial y}\right) \quad \begin{bmatrix} 11 \\ 4 \end{bmatrix} = \frac{\partial a_z}{\partial t} + \frac{\partial u}{\partial z} + i\left(\frac{\partial A_z}{\partial t} + \frac{\partial \phi}{\partial z}\right)$$

$$\begin{bmatrix} 12 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix} = \frac{1}{2}\left(\frac{\partial u}{\partial t} + i\frac{\partial \phi}{\partial t}\right) \quad \begin{bmatrix} 12 \\ 3 \end{bmatrix} = \frac{1}{2}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\begin{bmatrix} 14 \\ 2 \end{bmatrix} = \frac{1}{2}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \frac{1}{2}i\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \quad \text{etc.}$$

and

$$\left\{ \begin{matrix} \mu\nu \\ \alpha \end{matrix} \right\} = \begin{bmatrix} \mu\nu \\ \alpha \end{bmatrix} \text{ if } \alpha = 1, \quad \left\{ \begin{matrix} \mu\nu \\ \alpha \end{matrix} \right\} = -\begin{bmatrix} \mu\nu \\ \alpha \end{bmatrix} \text{ if } \alpha = 2, 3, 4.$$

Consider for instance the case $\nu = 2$. Equation (14) gives

$$(\rho + i\sigma)\frac{d^2x}{dt^2} = -(\rho + i\sigma)\left[\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} + 2\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}\frac{dx}{dt} + 2\left\{ \begin{matrix} 13 \\ 2 \end{matrix} \right\}\frac{dy}{dt} + 2\left\{ \begin{matrix} 14 \\ 2 \end{matrix} \right\}\frac{dz}{dt}\right]$$

$$(16) \qquad = (\rho + i\sigma)\left[\left\{\frac{\partial a_x}{\partial t} + \frac{\partial u}{\partial x} + \dot{x}\frac{\partial u}{\partial t} + \dot{y}\left(\frac{\partial a_x}{\partial y} - \frac{\partial a_y}{\partial x}\right) + \dot{z}\left(\frac{\partial a_z}{\partial z} - \frac{\partial a_z}{\partial x}\right)\right\}\right.$$

$$\left. + i\left\{\frac{\partial A_x}{\partial t} + \frac{\partial \phi}{\partial x} + \dot{x}\frac{\partial \phi}{\partial t} + \dot{y}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) + \cdot\left(\frac{\partial A_z}{\partial z} - \frac{\partial A_x}{\partial x}\right)\right\}\right]$$

If we introduce a quantity H defined by

$$(17) \qquad (H_x, H_y, H_z) = \text{curl } A$$

where $A = (A_x, A_y, A_z)$, and the quantity E defined by

$$(E_x, E_y, E_z) = -\dot{A} - \text{grad } \phi \tag{18}$$

and the cross product $[vH]$, then H, E and $[vH]$ behave as vectors, and the right-hand member of equation (16) becomes

$$(\rho + i\sigma)\frac{d^2x}{dt^2} = (\rho + i\sigma)\left\{\dot{a}_x + \frac{\partial u}{\partial x} + \dot{x}\frac{\partial u}{\partial t} - [vh]_x + i\left(-E_x - [vH]_x + \dot{x}\frac{\partial \phi}{\partial t}\right)\right\} \tag{19}$$

the real part of which is

$$\left\{\rho\frac{d^2x}{dt} = \rho\frac{\partial u}{\partial x} + \sigma(E_x + [vH]_x)\right\} + \rho(\dot{a}_x - [vh]_x) + \rho\dot{x}\frac{\partial u}{\partial t} - \sigma\dot{x}\frac{\partial \phi}{\partial t} \tag{20}$$

The first term in brackets is the classical law of motion of a charged mass particle under the influence of gravitational and electromagnetic forces; the other terms are in the nature of small corrections.

It is seen from (11) that the vectors E and H are derived from the retarded potentials in the same way as the electric and magnetic vectors of Maxwell's theory.

3. We conclude by examining the rôle of the Lorentz transformation in the present theory. To begin with we note that in the present theory Maxwell's equations and Lorentz's law of force are not given a priori but deduced from fundamental geometrical postulates in the same way as Newton's law of gravitation was deduced by Einstein in 1916, to the same order of approximation. If now there are two observers in motion with respect to each other we might inquire how far the approximate methods used in this paper for the derivation of Maxwell's

equations and Lorentz's law of force are valid. From (5) it is seen that the present treatment is legitimate only if the two systems of reference are connected with each other through a transformation which keeps the operator $\Box$ invariant, i.e. the Lorentz transformation.

It may be objected that Maxwell's equations are not in tensor form, i.e. that they are now on the same footing as Newton's law of gravitation. We may point out, however, that although we know that Maxwell's equations are covariant with respect to the Lorentz transformation provided that we assume that the electric and magnetic vectors are the components of an antisymmetric tensor, this assumption is not required by experimental evidence. It is well known that Michelson's experiment can be accounted for by the requirement that the light equation $\Box\, 4 = 0$ shall be invariant, i.e. that the two frames of reference are connected by the Lorentz transformation.

In concluding, I wish to thank Professor A. S. Eddington for his encouragement, and Professor M. S. Vallarta for his assistance in the preparation of this manuscript.

THE CAPACITY OF A LONG WIRE

By F. H. Murray

The object of this note is the approximate calculation of the static linear charge density on a long isolated wire, in which quantities of the order of $2\pi r/l$ are neglected, r being the radius and l the length of the wire. With this approximation and certain formulas already obtained, (1) the Fourier coefficients of the charge density may be determined without difficulty by a method of successive approximations, if the number of non-zero coefficients is assumed small, and the quantity $\alpha = \log (2l/r)$ is large. Results are tabulated for the first five Fourier coefficients, with values of α in the range $6 \leqslant \alpha \leqslant 14$. These results explain the success of the approximate method of Howe (2) for computing the capacity of an isolated wire.

1. *The Equations for the Fourier Coefficients.* Assume the potential to be a potential of a single layer, of which the density is constant over a circular cross-section; if $\sigma(x)$ is the linear density, the potential $P(x_0)$ is given by the formula[1]

$$P(x_0) = \int_0^l G(x - x_0)\, \sigma(x)\, dx$$

in which

$$G(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{w^2 + 2r^2 (1 - \cos \theta)}}$$

Let σ be represented by the series

$$\sigma(x) = \sum_{n=0}^{\infty} a_n \cos n (2\pi x/l)$$

[1] F. H. Murray, On the numerical calculation of the current in an antenna, *American Journal of Mathematics*, vol. 53, p. 886, (1931).
G. W. O. Howe, *The Electrician*, 73, p. 829, (1914); other applications of this method in 73, pp. 859, 906; 75, p. 870.

and assume an expansion in the same functions for the potential:

$$(1) \qquad P(x_0) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos n\,(2\pi\, x_0/l)$$

$$(2) \qquad b_m = \frac{2}{l}\int_0^l P(x_0) \cos m\,(2\pi\, x_0/l)\, dx_0$$

Or,

$$(3) \qquad b_m = \sum_{n=0}^{\infty} A_{mn}\, a_n,\ m = 0, 1, 2, \ldots.$$

where

$$A_{mn} = \frac{1}{2\,l}\int_0^l dx_0 \int_0^l dx\, G(x - x_0)\,[e^{a_m x_0 + a_n x} + e^{a_m x_0 - a_n x} + e^{-a_m x_0 + a_n x} + e^{-a_m x_0 - a_n x}]$$

$$= \frac{1}{2\,l}[I^{m,n} + I^{m,-n} + I^{-m,n} + I^{-m,-n}].$$

and

$$I^{m,n} = \int_0^l dx_0 \int_0^l dx\, G(x - x_0)\, e^{a_m x_0 + a_n x}$$

$$= \frac{1}{a_m + a_n}\{e^{l(a_m + a_n)}\,[I_{-n}(l) + I_{-m}(l)] - [I_m(l) + I_n(l)]\},\ a_m + a_n \neq 0,$$

Also,

$$I^{m,-m} = l\,[I_m(l) + I_{-m}(l)] - \frac{dI_m(l)}{da_m} + \frac{dI_{-m}(l)}{da_m},$$

$$I_m(l) = \int_0^l G(w)\, e^{a_m w}\, dw = \log\frac{2\,l}{r} - y - \log(l|a_m|) + Ci\,(l|a_m|) + \frac{a_m}{|a_m|}\, Si\,(l|a_m|),\ (y = .5772157).$$

The constants A_{0n} are found by taking limits; the results are,

$$-A_{m,n} = \frac{Si\,(2m\pi) + Si\,(2n\pi)}{(m+n)\pi} + \frac{Si\,(2m\pi) - Si\,(2n\pi)}{(m-n)\pi}, \quad m \neq n,$$

$$A_{m,m} = 2\left[\log\frac{2l}{r} - y - \log(2m\pi) + Ci\,(2m\pi)\right] - Si\,(2m\pi)/m\pi, \quad m \neq 0. \tag{4}$$

$$-A_{0,n} = 2\,Si\,(2n\pi)/n\pi, \quad n \neq 0,$$

$$A_{0,0} = 4\left[\log\frac{2l}{r} - 1\right].$$

Since the potential must be constant along the wire, $b_1 = b_2 = b_3 \ldots = 0$; the equations (3) can then be thrown into the form

$$\begin{aligned} a_0 &= (b_0/A_{00}) - (A_{01}/A_{00})\,a_1 - (A_{02}/A_{00})\,a_2 - (A_{03}/A_{00})\,a_3 - \ldots. \\ a_1 &= -(A_{10}/A_{11})\,a_0 - \quad -(A_{12}/A_{11})\,a_2 \quad -(A_{13}/A_{11})\,a_3 - \ldots, \\ a_2 &= -(A_{20}/A_{22})\,a_0 - (A_{21}/A_{22})\,a_1 \quad -(A_{23}/A_{22})\,a_3 - \ldots. \end{aligned} \tag{5}$$

. . . . . .

To solve these, set up the system

$$\begin{aligned} a_0^{(n+1)} &= (b_0/A_{00}) - (A_{01}/A_{00})\,a_1^{(n)} - (A_{02}/A_{00})\,a_2^{(n)} - \ldots. \\ a_1^{(n+1)} &= -(A_{10}/A_{11})\,a_0^{(n)} \quad -(A_{12}/A_{11})\,a_2^{(n)} - \ldots. \\ a_2^{(n+1)} &= -(A_{20}/A_{22})\,a_0^{(n)} - (A_{21}/A_{22})\,a_1^{(n)} - \ldots. \end{aligned} \tag{6}$$

. . . . . .

$$n = 0, 1, 2 \ldots.$$

and let $a_0^0 = (b_0/A_{00})$, $a_n^0 = 0$, $n = 1, 2, 3 \ldots$. If $a = \log(2l/r) \geqslant 6$, the ratios A_{mn}/A_{nn} are small, and if the number of equations is small the approximations converge rapidly; the

approximation is still more rapid if in calculating $a_m^{(n)}$, the calculated values of $a_0^{(n)}$, $a_1^{(n)}$, $a_{m-1}^{(n)}$ are substituted on the right.

If five equations in the first five Fourier coefficients are set up, the results are given in the table below, with $A = b_0/A_{00} = P/2\,[\log(2l/r) - 1]$

a	a_0/A	a_1/A	a_2/A	a_3/A	a_4/A
6	1.011	0.15	0.10	0.082	0.071
8	1.0045	0.088	0.055	0.041	0.034
10	1.0025	0.063	0.037	0.027	0.022
12	1.0016	0.049	0.029	0.020	0.016
14	1.0011	0.041	0.023	0.016	0.013

2. *Discussion and Conclusions.* Since the coefficients of the equations $-A_{mn}/A_{mm}$ are all positive, if $A_{mm} > 0$, the calculated values of $a_1, a_2, \ldots$ are positive also; the charge density σ at the ends of the wire, equal to the sum $A\Sigma a_n$ will be greater than the sum of the first m terms by an unknown amount; in fact, it is easy to see that if m, the number of equations, is so small that $A_{mm} > 0$, the calculated value of $a_m^{(n)}$ increases with n, and

$$\sum_{n=0}^{m} a_n > \sum_{n=0}^{m} a_n^{(k)} > \sum_{n=0}^{m} a_n^1 = \sum_{n=0}^{m} (-A_{on}/A_{nn})$$

and

$$-A_{on} \sim \frac{1}{n}, \quad A_{nn} \sim 2\left[\log \frac{l}{n\pi r} - y\right]$$

The series

$$\sum \frac{1}{n\,[\log n - a]} = \sum \frac{c}{cn \log cn}, \quad -a = \log c,$$

diverges; it cannot be concluded, however, that $\sigma = \infty$ at the ends of the wire since the coefficients A_{mn} can be computed from (4) only if $2m\pi r/l \ll 1$, $2n\pi r/l \ll 1$.

The capacity of the wire is given by

$$C = \frac{la_0}{P} = \frac{a_0}{A} \cdot \frac{l}{2[\log (2l/r) - 1]} e.s.u.$$

and the numerical values in the first column of the table indicate the error to be expected if (a_0/A) is replaced by 1, which corresponds to solving the first equation with only $a_0 \neq 0$. This error is less than one per cent if $a > 8$; that is, within an error of one per cent the capacity may be computed by determining the average potential due to a constant charge distribution along the wire, and assuming $C = la_0/P_{av}$, which is the method of Howe.

ON ORTHOGONAL MATRICES

By R. E. A. C. Paley

1. The results of this paper have application in the theory of polytopes. They are referred to in the two papers immediately following this one, by Dr. Todd and Dr. Coxeter, who suggested the problem to me, and showed me the proof of Lemma 2.

We investigate the conditions under which we may construct a square orthogonal[1] matrix with all its elements ± 1. Such matrices have been considered as long ago as 1867 by Sylvester,[2] who gives an explicit construction for the case when m is of the form 2^k. Let m be the order of the matrix. We observe first that, apart from the exceptional cases[3] $m = 1,2$, it is necessary that m should be divisible by 4. For let the matrix be represented by

$$A_{i,j} \qquad (0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant m-1; m \geqslant 3).$$

Then

$$\sum_{j=0}^{m-1} (A_{1,j} + A_{2,j})(A_{1,j} + A_{3,j}) = \sum_{j=0}^{m-1} A_{1,j}^2 = m.$$

[1] We call a square matrix

$$A_{i,j} \ (0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant m-1)$$

orthogonal, if, whenever i_1 and i_2 are unequal,

$$\sum_{j=0}^{m-1} A_{i_1,j} A_{i_2,j} = 0.$$

This is known to be equivalent to the condition

$$\sum_{i=0}^{m-1} A_{i,j_1} A_{i,j_2} = 0 \qquad (j_1 \neq j_2).$$

[2] Sylvester 6.

[3] For $m = 2$, the matrix is, of course,

$$\begin{matrix} +1 & +1 \\ +1 & -1 \end{matrix}$$

Since $(A_{1,j} + A_{2,j})(A_{1,j} + A_{3,j}) = 4$ or 0, we see that it must be 4 for $\frac{1}{4}m$ values of j, and that m must be divisible by 4. It seems probable that, whenever m is divisible by 4, it is possible to construct an orthogonal matrix of order m composed of ± 1, but the general theorem has every appearance of difficulty. We can however find quite simply a few cases in which it is certainly possible to construct such a matrix. We will call a matrix of the kind considered a U-matrix, and, in writing out a U-matrix, we will write $+$ and $-$ to denote $+1$ and -1.

2. We state the results we obtain as lemmas.

LEMMA 1. *If we have a U-matrix of order m_1, and a U-matrix of order m_2, then we may construct a U-matrix of order $m_1 m_2$.*

Let M_1 denote the matrix of order m_1, and let $-M_1$ denote the same matrix with $+$ and $-$ interchanged. Let M_2 denote the matrix of order m_2. Now for each $+$ of M_2 we substitute a matrix M_1, and for each $-$ a matrix $-M_1$. The resulting square matrix is of order $m_1 m_2$, and is orthogonal. The proof is immediate.

When $m_1 = 2$ and $m_2 = 4$, or *vice versa*, we have

$$\begin{array}{cccccccc}
+ & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + \\
+ & + & + & + & - & - & - & - \\
+ & - & + & - & - & + & - & + \\
+ & + & - & - & - & - & + & + \\
+ & - & - & + & - & + & + & -
\end{array}$$

LEMMA 2. *Let m be of the form $p + 1$, where p, $\equiv 3 \pmod 4$, is prime. Then we can construct a U-matrix of order m.*

Let $\chi(n)$ denote the Legendre symbol (n/p). We write

$$\begin{aligned}
A_{i,j} &= +1 && (i = 0 \text{ or } j = 0), \\
A_{i,j} &= \chi(j - i) && (1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant p,\ i \neq j), \\
A_{i,i} &= -1 && (1 \leqslant i \leqslant p).
\end{aligned}$$

Then, if i_1 and i_2 are different and both greater than 0, we have

$$(2.1)\qquad \sum_{j=0}^{p} A_{i_1,j} A_{i_2,j} = A_{i_1,i_1} A_{i_2,i_1} + A_{i_1,i_2} A_{i_2,i_2} + A_{i_1,0} A_{i_2,0} + \sum_{j=1}^{p} \chi(j - i_1)\, \chi(j - i_2).$$

Now the first two terms on the right hand side of (2.1) are

$$-\chi(i_1 - i_2) - \chi(i_2 - i_1),$$

and are equal and opposite, since $p \equiv 3 \pmod 4$ and thus $\chi(-1) = -1$. The third term is $+1$, and the last sum

$$\sum_{j=1}^{p} \chi(j - i_1)\, \chi(j - i_2)$$

is known[4] to be -1. Thus we have

$$\sum_{j=0}^{p} A_{i_1,j} A_{i_2,j} = 0 \quad (0 \neq i_1 \neq i_2 \neq 0).$$

Finally, if $i \neq 0$,

$$\sum_{j=0}^{p} A_{0,j} A_{i,j} = \sum_{j=0}^{p} A_{i,j} = A_{i,i} + A_{i,0} + \sum_{j=1}^{p} \chi(j - i) = 0.$$

Thus the matrix $A_{i,j}$ is orthogonal, and our lemma is proved.

If $m = 12$, the method of Lemma 2 gives the matrix

$$\begin{array}{cccccccccccc}
+ & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & + & + & - & - & - & + & - \\
+ & - & - & + & - & + & + & + & - & - & - & + \\
+ & + & - & - & + & - & + & + & + & - & - & - \\
+ & - & + & - & - & + & - & + & + & + & - & - \\
+ & - & - & + & - & - & + & - & + & + & + & - \\
+ & - & - & - & + & - & - & + & - & + & + & + \\
+ & + & - & - & - & + & - & - & + & - & + & + \\
+ & + & + & - & - & - & + & - & - & + & - & + \\
+ & + & + & + & - & - & - & + & - & - & + & - \\
+ & - & + & + & + & - & - & - & + & - & - & + \\
+ & + & - & + & + & + & - & - & - & + & - & -
\end{array}$$

[4] I am indebted to Mr. H. Davenport who pointed out the relevance of this equation.

LEMMA 3. *Let m be divisible by 4 and of the form $2^k(p+1)$, where p is prime. Then we can construct a U-matrix of order m.*

If $p = 2$, or if $p \equiv 3 \pmod 4$, the result follows at once from Lemmas 1 and 2. We may thus assume that $p \equiv 1 \pmod 4$, $k = 1$.

We write

$$A_{2i,0} = A_{2i,1} = A_{2i+1,0} = -A_{2i+1,1} = 1 \quad (1 \leqslant i \leqslant p),$$

$$A_{0,2i} = A_{0,2i+1} = A_{1,2i} = -A_{1,2i+1} = 1 \quad (1 \leqslant i \leqslant p),$$

$$A_{2i,2j} = A_{2i,2j+1} = A_{2i+1,2j} = -A_{2i+1,2j+1} = \chi(j-i) \qquad (1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant p,\ i \neq j),$$

$$A_{2i,2i} = -A_{2i,2i+1} = -A_{2i+1,2i} = -A_{2i+1,2i+1} = 1 \quad (0 \leqslant i \leqslant p).$$

This is equivalent to taking the $(p+1)$-rowed matrix

$$B_{i,0} = B_{0,i} = 1 \qquad (1 \leqslant i \leqslant p),$$

$$B_{i,j} = \chi(j-i) \qquad (1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant p,\ i \neq j),$$

$$B_{i,i} = 0 \qquad (0 \leqslant i \leqslant p)$$

and then substituting the matrices

$$\begin{matrix} + & + \\ + & - \end{matrix}, \quad \begin{matrix} - & - \\ - & + \end{matrix}, \quad \begin{matrix} + & - \\ - & - \end{matrix}$$

respectively for $+1$, -1 and 0 in the matrix $B_{i,j}$.

To prove that the matrix $A_{i,j}$ defined above is a U-matrix, we observe first that the matrix $B_{i,j}$ is orthogonal by the same argument that was used in Lemma 2. Now we write

$$A_{i,j} = C_{i,j} + D_{i,j} \qquad (0 \leqslant i \leqslant 2p+1,\ 0 \leqslant j \leqslant 2p+1),$$

where $C_{i,j}$ is the result of substituting

$$\begin{matrix} + & + & 0 & 0 \\ + & - & 0 & 0 \end{matrix}, \quad \begin{matrix} - & - & 0 & 0 \\ - & + & 0 & 0 \end{matrix}$$

respectively for $+1$, -1, and 0 in the matrix B, and $D_{i,j}$ is the result of substituting

$$\begin{matrix} 0\ 0 \\ 0\ 0 \end{matrix}' \quad \begin{matrix} 0\ 0 \\ 0\ 0 \end{matrix}' \quad \begin{matrix} + & - \\ - & - \end{matrix}$$

Now, if $i_1 \neq i_2$, we have

$$\sum_{j=0}^{2p+1} A_{i_1,j} A_{i_2,j} = \sum_{j=0}^{2p+1} (C_{i_1,j} C_{i_2,j} + D_{i_1,j} D_{i_2,j}) + \sum_{j=0}^{2p+1} (C_{i_1,j} D_{i_2,j} + D_{i_1,j} C_{i_2,j}). \tag{2.2}$$

The first sum on the right hand side of (2.2) is easily seen to vanish. The second certainly vanishes except when one of i_1, i_2 is even and the other odd. Suppose then that i_1 is even and i_2 odd, and that they are both greater than 1. Then

$$\sum_{j=0}^{2p+1} C_{i_1,j} D_{i_2,j} = C_{i_1,i_2} D_{i_2,i_2} + C_{i_1,i_2-1} D_{i_2,i_2-1} = -2\chi(\tfrac{1}{2}(i_2 - 1) - \tfrac{1}{2}i_1).$$

Similarly

$$\sum_{j=0}^{2p+1} D_{i_1,j} C_{i_2,j} = 2\chi(\tfrac{1}{2}i_1 - \tfrac{1}{2}(i_2 - 1)).$$

Thus the second sum on the right hand side of (2.2) is

$$-2\{\chi(\tfrac{1}{2}(i_2 - 1) - \tfrac{1}{2}i_1) - \chi(\tfrac{1}{2}i_1 - \tfrac{1}{2}(i_2 - 1))\},$$

and vanishes since $p \equiv 1 \pmod 4$ and thus $\chi(-1) = 1$. If i_1 or i_2 is 0 or 1 the sum may similarly be proved to vanish. Thus, for $i_1 \neq i_2$, we have proved that

$$\sum_{j=0}^{m-1} A_{i_1,j} A_{i_2,j} = 0;$$

and this establishes Lemma 3.

LEMMA 4. *Let m be divisible by 4 and of the form $2^k(p^k + 1)$ where p is an odd prime. Then we can construct a U-matrix of order m.*

We may repeat the argument of the last two lemmas, but instead of quadratic residues (mod. p) we consider quadratic residues in the Galois field[5] of polynomials (mod. p, mod. $P(x)$), where $P(x)$ is an irreducible polynomial of degree h. Thus, for instance, if $p^h \equiv 3 \pmod 4$, $k = 0$, we write

$$A_{i,j} = +1 \qquad (i = 0 \text{ or } j = 0),$$

$$A_{i,j} = \chi(\xi_j - \xi_i) \qquad (1 \leqslant i \leqslant p^h,\ 1 \leqslant j \leqslant p^h,\ i \neq j),$$

$$A_{i,i} = -1 \qquad (1 \leqslant i \leqslant p^h),$$

where $\xi_1, \xi_2, \ldots, \xi_{p^h}$ denote the marks of the field arranged in any order. If $m = 28 = 3^3 + 1$ and $P(x) = x^3 + 2x + 1$, for instance, we get the matrix

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[5] For the definition of Galois fields and their properties see Dickson, 1, 3–54.

LEMMA 5. *Let m be of the form $2^k p(p+1)$, where p, $\equiv 3$ (mod. 4), is prime. Then we can construct a U-matrix of order m.*

This follows by Lemma 2 from a result due to Scarpis[6] who shows that if p is prime, and we can construct a U-matrix of order $p+1$, then we can construct such a matrix of order $2^k p(p+1)$.

(The least of such values of m which is not covered by Lemma 4 is $43.44 = 1892$.)

LEMMA 6. *If m, $\leqslant 200$, is divisible by 4, we may construct a U-matrix except possibly in the cases $m = 92, 116, 156, 172, 184, 188$.*

We may tabulate the different cases as shown in Table 1. Where more than one solution exists we give one only.

TABLE 1

$m = 4 = 2^2$	$m = 104 = 2^3(5^2+1)$
$m = 8 = 2^3$	$m = 108 = 107+1$
$m = 12 = 11+1$	$m = 112 = 2^3(3^3+1)$
$m = 16 = 2^4$	$m = 116$
$m = 20 = 19+1$	$m = 120 = 2(59+1)$
$m = 24 = 2(11+1)$	$m = 124 = 2(61+1)$
$m = 28 = 3^3+1$	$m = 128 = 2^7$
$m = 32 = 2^5$	$m = 132 = 131+1$
$m = 36 = 2(17+1)$	$m = 136 = 2^3(37+1)$
$m = 40 = 2(19+1)$	$m = 140 = 139+1$
$m = 44 = 43+1$	$m = 144 = 2^3(17+1)$
$m = 48 = 2^3(11+1)$	$m = 148 = 2(73+1)$
$m = 52 = 2(5^2+1)$	$m = 152 = 2^3(37+1)$
$m = 56 = 2(3^3+1)$	$m = 156$
$m = 60 = 59+1$	$m = 160 = 2^3(19+1)$
$m = 64 = 2^6$	$m = 164 = 163+1$
$m = 68 = 67+1$	$m = 168 = 2(83+1)$
$m = 72 = 2^3(17+1)$	$m = 172$
$m = 76 = 2(37+1)$	$m = 176 = 2^3(43+1)$
$m = 80 = 2^3(19+1)$	$m = 180 = 2(89+1)$
$m = 84 = 83+1$	$m = 184$
$m = 88 = 2(43+1)$	$m = 188$
$m = 92$	$m = 192 = 2^4(11+1)$
$m = 96 = 2^3(11+1)$	$m = 196 = 2(97+1)$
$m = 100 = 2(7^2+1)$	$m = 200 = 2^3(7^2+1)$

3. We can assert more in the case where m is a power of 2. We prove the following theorem:

[6] Scarpis, 5.

THEOREM. *Let m be a power of 2. Then we can arrange the 2^m different rows of m + 's and −'s in $2^m/m$ U-matrices, so that each row is contained (as a row) by exactly one of the matrices.*

Let $m = 2^k$. Let

$$i = \sum_{\lambda=0}^{k-1} \eta_\lambda 2^\lambda \qquad (0 \leqslant i \leqslant 2^k - 1),$$

$$j = \sum_{\lambda=0}^{k-1} \zeta_\lambda 2^\lambda \qquad (0 \leqslant j \leqslant 2^k - 1),$$

where all the numbers η_λ, ζ_λ $(0 \leqslant \lambda \leqslant k-1)$ are 0 or 1. Then we write

$$A_{i,j} = \prod_{\lambda=0}^{k-1} (-1)^{\eta_\lambda \zeta_{k-1-\lambda}}$$

We denote by M the matrix[7] so defined. (M is in fact the product, in the sense of Lemma 1, of the matrix

$$\begin{matrix} + & + \\ + & - \end{matrix}$$

by itself k times).

We observe first that, if

$$i_1 = \sum_{\lambda=0}^{k-1} \eta_\lambda^{(1)} 2^\lambda, \qquad i_2 = \sum_{\lambda=0}^{k-1} \eta_\lambda^{(2)} 2^\lambda,$$

where $\eta_\lambda^{(1)}$ and $\eta_\lambda^{(2)}$ are 0 or 1 $(0 \leqslant \lambda \leqslant k-1)$, then

$$A_{i_1,j} A_{i_2,j} = A_{i,j} \tag{3.1}$$

where

$$i = \sum_{\lambda=0}^{k-1} \eta_\lambda 2^\lambda,$$

$$\eta_\lambda = 0 \text{ or } 1, \qquad \eta_\lambda \equiv \eta_\lambda^{(1)} + \eta_\lambda^{(2)} \pmod 2,$$

[7] $A_{i,j}$ is in fact the value of the ith Walsh function in the interval $j/m \leqslant k < (j+i)/m$. For the properties of these functions see Walsh, 7, Kaczmarz, 3, Paley, 4.

and i is different from zero except when $i_1 = i_2$. Thus the rows

$$\{A_{i,j}\} \qquad (0 \leqslant j \leqslant 2^k - 1),$$

for $i = 0, 1, \ldots 2^k - 1$, form a group for term by term multiplication. Further, if $i_1 \neq i_2$, (3.1) gives

$$\sum_{j=0}^{m-1} A_{i_1,j} A_{i_2,j} = \sum_{j=0}^{m-1} A_{i,j} = 0,$$

since i does not vanish. Thus the matrix $A_{i,j}$ is orthogonal.

Now let $B_0, B_1, \ldots, B_{m-1}$ denote any ordered array of ± 1's. Corresponding to the array B we have an orthogonal matrix

$$(3.2) \qquad A_{i,j} B_j \qquad (0 \leqslant i \leqslant m - 1, 0 \leqslant j \leqslant m - 1),$$

$A_{i,j}$ denoting the elements of the matrix M. There are 2^m matrices of the form (3.2). Let M_1, M_2 be two of them, and suppose that M_1, M_2 have a row in common, so that, if $B_j^{(1)}$ $(0 \leqslant j \leqslant m - 1)$ and $B_j^{(2)}$ $(0 \leqslant j \leqslant m - 1)$ denote the first rows of M_1 and M_2 respectively, then, for appropriate i_1, i_2, we have

$$A_{i_1,j} B_j^{(1)} = A_{i_2,j} B_j^{(2)} \qquad (0 \leqslant j \leqslant m - 1).$$

Consider the $(i + 1)$st row

$$A_{i,j} B_j^{(1)} \qquad (0 \leqslant j \leqslant m - 1)$$

of M_1. We have

$$(3.3) \qquad A_{i,j} B_j^{(1)} = A_{i,j} A_{i_1,j} A_{i_2,j} B_j^{(2)} \; (0 \leqslant j \leqslant m - 1).$$

In virtue of (3.1) the right hand side of (3.3) is a row of M_2. Thus each row of M_1 is a row of M_2 and conversely. It follows that the 2^m matrices (3.2) can be divided into $2^m/m$ sets of m, such that, if M_1 and M_2 belong to the same set, then every row of M_1 is a row of M_2 and conversely, while, if M_1 and M_2 belong to different sets, they have no rows in common. We have only

to choose one matrix out of each of the $2^m/m$ sets and our theorem is proved.

4. Since writing this paper I have discovered that some of the results of the paper have been stated by Gilman.[8] Professor Gilman tells me that the arguments I use in Lemmas 1 and 2 are substantially the same as his.

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[8] Gilman, 2.

A COMBINATORIAL PROBLEM

By J. A. Todd

1. The present paper is concerned primarily with the following combinatorial problem, which arises in connection with the theory of polytopes, and is referred to in the paper by Dr. Coxeter which succeeds this one. The problem is:

Given $4n-1$ objects, to find $4n-1$ sets, each containing $2n-1$ of the objects, such that any pair of sets have exactly $n-1$ common members.

The problem is equivalent to finding an orthogonal matrix of $4n$ rows and columns every element of which is either $+1$ or -1. For consider such a matrix. By changing the signs of the elements of a certain number of rows and columns it may be reduced to a standard form in which every element in the first row and column is $+1$. Then the condition of orthogonality requires that in the matrix of $4n-1$ rows and columns obtained by suppressing the first row and column, each row should contain $2n-1$ elements $+1$, and any two rows should have just $n-1$ pairs of elements $+1$ which occur in the same column. Conversely, it is clear that from every arrangement of $4n-1$ objects in sets in the manner described we can deduce a square orthogonal matrix.

In the preceding paper[1] Mr. Paley has given general rules for constructing such a matrix when $4n$ is a number belonging to one of certain special types. The present paper deals in detail with the simpler cases, and shows incidentally that these general methods by no means exhaust the possibilities. We shall also consider, in these simple cases, the group of permutations (of the objects) under which the arrangement in sets remains invariant.

2. Consider the orthogonal matrix M of 2^k rows and columns defined in §3 of Paley's paper. It is there shown that the rows

[1] Paley, 4.

of this matrix form a group for term by term multiplication. Every operation of this group is of period two, and it follows that the group is the Abelian group of order 2^k and type $(1, 1, \ldots, 1)$ generated by k independent operations of period two. Corresponding to the matrix M is a solution of the combinatorial problem where $n = 2^{k-2}$. A permutation of the elements which leaves such a solution unchanged corresponds to a permutation of the columns of M (excluding the first) which has the effect of changing M into a new matrix differing from the original one only in the order of the rows. Such a permutation is an automorphism of the Abelian group, and thus the group of permutations which leaves the arrangement in sets fixed is the group of automorphisms of the Abelian group; that is, it is the linear homogeneous group of order

$$(2^k - 1)(2^k - 2)(2^k - 2^2) \ldots . (2^k - 2^{k-1}).$$

We may give a geometrical interpretation to this. Consider the finite projective geometry of $k - 1$ dimensions,[2] $PG(k - 1, 2)$, in which each point is specified by a set of k numbers, reduced mod. 2, the zero set being excluded. There are $2^k - 1$ such points. Any homogeneous linear relation between the coordinates picks out a set of $2^{k-1} - 1$ points forming a space of $k - 2$ dimensions S_{k-2} (in the sense of the finite geometry), and the points common to two such spaces are $2^{k-2} - 1$ in number. Evidently, if we regard the points as objects and the sets of points lying in the S_{k-2} as sets, we have a solution of the combinatorial problem; and it is not difficult to see that this is the same as that derived from the matrix M. The linear homogeneous group then appears as the collineation group in the finite geometry.

For $k = 3$ we have the well-known arrangement of seven symbols in seven triads, any two of which have a common member, the whole being invariant under the simple group of order 168.[3]

[2] Veblen and Bussey, 6, 244.
[3] Burnside, 2, 216.

For $k = 4$ we have an arrangement of fifteen symbols in sets of seven, any two sets having three common symbols, invariant under a group of order 20160 simply isomorphic with the alternating group of degree eight.[4] We shall see later that there are other arrangements of fifteen symbols in sets of this kind, invariant under different groups, and thus fundamentally distinct from the arrangement here considered.

3. Let us now suppose $4n-1$ to be a prime p. By Lemma 2 of Paley's paper we can obtain an arrangement of the numbers $0,1, \ldots p-1$ in sets satisfying the required conditions by taking for one set the set of quadratic residues (mod. p) and forming the other sets by adding to each element of this set $1,2, \ldots, p-1$ and reducing mod. p. Thus, for $p = 7$, we get the arrangement

124, 235, 346, 450, 561, 602, 013,

which is invariant under the G_{168} generated by

(1234560), (124) (653), (45) (1632).

We pass on to the case $p = 11$, and for convenience replace 10 by t. The sets are now

13459, 2456t, 35670, 46781, 57892, 689t3, 79t04, 8t015, 90126, t1237, 02348.

We can easily show that there is no arrangement of eleven symbols in sets satisfying the conditions of the problem which is not equivalent to the above. For, denote the symbols by the letters *abcdefghjkl*, and suppose *abcde*, *abfgh* to be two of the sets, having in common the pair *ab*. If the pair *ab* occurred in a third set this set would have to be *abjkl*, and there is no set of five letters having two only in common with each of the three sets containing *ab*. Thus *ab* cannot occur together again, and the ten sets other than *abcde* each contain a different pair of these letters. We may thus take the sets involving *a* to be

abcde, abfgh, acfkl, adgjl, aehjk.

[4] *ibid.*, 456.

The set which contains bc cannot involve f, or it would have three symbols in common with $acfkl$. It contains either g or h, say g. Then the remaining sets can only be

$$bcgjk,\ bdhkl,\ befjl,\ cdfhj,\ ceghl,\ defgk.$$

There is thus only one possible type, and the one obtained by the rule is identified with it by making the substitution

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & t & 0 \\ a & g & b & c & e & l & f & k & d & h & j \end{matrix}$$

From the method by which the arrangement was constructed it is easily seen that eleven symbols can be arranged in this way in

$$\frac{1}{11}\binom{11}{5}\binom{6}{3}3!\,3!\,2 = \frac{11!}{660}$$

different ways, so that each such arrangement is invariant under a permutation group of order 660. We shall identify this with the known simple group of that order.

Among the permutations of the group are clearly the three following:

$$S_1 = (12345678t0),\ S_2 = (14593)\ (28t76),$$

$$S_3 = (195)\ (028)\ (67t).$$

and these in fact generate the group; for S_2 and S_3, which both keep fixed the first set, generate an icosahedral group, of order 60. We have, moreover,

$$S_2S_3 = (13)\ (2t)\ (45)\ (80),\ S_2S_1^4 = (27)\ (3t)\ (40)\ (89),$$

and

$$S_2S_1^4 = S_1^4S_2 = (1984t)\ (05367).$$

Thus if we put $S_1^4 = S$ we have $S^{11} = S_2^6 = S_3^2 = (S_2S_3)^2 = (S_2S)^2 = E$, $S_2S = S^4S_2$; and these relations do in fact form an abstract definition for the group G_{660}.[5]

It is evident that for any general value of p the solution obtained by the rule always admits automorphisms belonging to the group of order $\frac{1}{2}p(p-1)$ generated by the operations $X' = X + 1$ and $X' = a^2X$, where X, X' are reduced mod. p and a is a primitive root. When $p = 7$ or 11 we have seen that the whole group of automorphisms of the arrangement is of greater order than this. But when $p = 19$ this is no longer the case. For consider the set of residues 1,4,5,6,7,9,11,16,17. If we write out the table at length we find that each pair of these numbers occurs in three of the remaining eighteen sets, and four of the numbers occur in each set. Thus each pair of these numbers is associated with three other pairs. On examination we find that the six numbers involved in these pairs are distinct if, and only if, the original pair consists of a pair of adjacent numbers in the sequence 1,9,5,7,6,16,11,4,17,1, which will be recognized as the sequence of residues of powers of 9. Thus every operation of the group of automorphisms which keeps fixed the set of residues and the symbol 1 leaves this sequence invariant, and hence either leaves each of the residues unaltered or interchanges in pairs 9,17; 5,4; 7,11; 6,16. In the former case the operation is identity; in the latter case the operation cannot exist, for it would change the triad 1,4,6 (which occurs in three of the sets) into 1,5,16 (which occurs in two only). We conclude that the only operation of the group keeping fixed the set of residues and the symbol 1 is identity. The group of automorphisms is thus just the group of order 19.9 generated by the two operations which always exist in this type of solution.

It may similarly be verified that the solution when $p = 23$ admits only a group of order 23.11, but a discussion of the general case does not seem tractable. A similar remark applies to Paley's solution when $4n - 1$ is a prime power.[6] In the case

[5] Todd, **5**, 200.
[6] Paley, **4**, 316 (Lemma 4).

when $p = 3$ and $P(x) = x^3 + 2x + 1$ $(n = 7)$, the group is generated by the operations $X' = X + 1$, $X' = X + x$, $X' = X + x^2$ and $X' = x^2X$ (where X, X' are marks of the field). Its order therefore[7] is $3^3.13$.

4. If an arrangement of $4n - 1$ objects in sets be known, one of $8n - 1$ objects can be deduced as follows. Let $A_1, A_2, \ldots, A_{4n-1}$ be the sets formed from the symbols $a_1, a_2, \ldots, a_{4n-1}$; and let $B_1, B_2, \ldots, B_{4n-1}$ be the sets formed from the symbols $b_1, b_2, \ldots, b_{4n-1}$. It is not necessarily implied that the arrangement B is identical with the arrangement A. Denote by B'_i the set of symbols b obtained by omitting those which occur in B_i. If c is any other symbol, an arrangement of the $8n - 1$ symbols a, b, c is given by the sets

$$A_i B'_i \qquad (i = 1, 2, \ldots, 4n - 1),$$

$$cA_i B_i \qquad (i = 1, 2, \ldots, 4n - 1),$$

$$a_1 a_2 \ldots a_{4n-1}.$$

The verification is immediate. If the sets B_i are respectively the same permutations of b as A_i are of a, this is a particular case (the simplest) of Paley's Lemma 1, with $m_2 = 2$.

5. The last case we shall consider is the arrangement of fifteen objects in fifteen sets of seven, any two sets having a common triad. We shall denote the objects by the letters a, b, c, d, e, f, g, h, k, l, m, p, q, r, s. From the fact that any two of the sets have three common letters and no more, it is easy to see that each pair of the fifteen letters occurs in just three sets, and each letter in seven. Let us pick out one set, *abcdefg*. Each of the other fourteen sets contains three of these letters; every letter of the set occurs in six of these triads, and each pair of letters in two. A straightforward investigation shows that a set of fourteen triads of seven letters having this property must be of one of four types.

[7] It is easily verified that x is a primitive root of the equation $a^{26} = 1$.

(A) The fourteen triads may be all different, in which case they fall into two distinct sets of seven, each pair of a set having a common letter. Such a set is

abc, ade, afg, bdg, bef, cdf, ceg,

def, cfg, bcd, ace, adg, beg, abf.

(B) One of the triads may be repeated, in which case the triads can be arranged in two sets of seven such as

abc, ade, afg, bdg, bef, cdf, ceg,

abc, bfg, bde, cef, cdg, aeg, adf.

(C) Three of the triads may be repeated, in which case the set takes the form

abc, ade, afg, bdg, bef, cdf, ceg,

abc, ade, afg, cef, cdg, beg, bdf.

(D) The set may consist of seven triads each taken twice,

abc, ade, afg, bdg, bef, cdf, ceg.

Every set of seven letters in our arrangement must belong to one or other of these four types, and may be denoted by a symbol A, B, C or D as the case may be. Thus the aggregate of sets may be denoted by a symbol of the form $A^{\lambda} B^{\mu} C^{\nu} D^{\rho}$, where $\lambda + \mu + \nu + \rho = 15$. We must now determine the possible values of λ, μ, ν, ρ.

It is evident that if one set be of type D every other set is of type B, C or D, since it contains at least one triad which occurs in all in three sets. If two sets are of type D it is easy to see that the third set which contains their common triad is of the same type. In this case every other set contains three repeated triads at least, and is thus of type C or D. Finally, if there be more than three sets D every set must be of this type. Hence we have for ρ the possible values 0, 1, 3, 15; and, if $\rho = 3$, $\nu = 12$.

Suppose now there is only one set of type D. The other sets are of type B or C. If one of the sets be of type C, it has three repeated triads, one being common to it and the set of type D; and these three triads all contain a common letter. It is then easily verified that the remaining sets which contain this letter must also be of type C, and that no other sets can be of this type. Hence, if $\rho = 1$, we have either $\nu = 6$, $\mu = 8$ or $\nu = 0$, $\mu = 14$.

Now suppose there are no sets of type D. If *abcdefg* is of type B, with *abc* for repeated triad, let *abchklm*, *abcpqrs* be the other sets containing *abc*, and suppose that the triads of *abcdefg* occurring in the other sets are the ones previously written down. If we try to complete the sets involving the triads *ade*, *afg*, *bde*, *bfg*, we find that either *abchklm* or *abcpqrs* has a pair of repeated triads in common with these; and a similar remark applies to the sets involving *bef*, *bdg*, *cef*, *cdg* or *adf*, *aeg*, *cdf*, *ceg*. We are thus led to a set of type D. Hence, if $\rho = 0$, we must have $\mu = 0$ also, and A and C are the only possible types. If one of the sets be of type C, with three repeated triads, all the other sets containing one of these triads must also be of type C, since by what has been shown they cannot be of type B. We have thus seven sets of type C, and it is soon seen that these are just the seven sets involving one letter. There cannot be more than seven sets of this type. Finally, suppose, if possible, that all the sets were of type A, and that the triads of *abcdefg* were the ones previously given. Then if we take as two further sets *abchklm* and *deghklp* we soon find that three of the sets contain the triad *mpf*, and are not of type A. This case is therefore impossible. Hence, if $\rho = 0$ we have $\nu = 7$, $\lambda = 8$.

Thus, finally we arrive at the conclusion that every possible arrangement of the fifteen letters is of one of the five types

$$D^{15},\ C^{12}\,D^{3},\ B^{8}\,C^{6}\,D,\ B^{14}\,D,\ A^{8}\,C^{7}.$$

The actual existence of arrangements of each of these types will be demonstrated by giving a pattern of each type.

D^{15}: *abcdefg*, *abchklm*, *abcpqrs*, *adehkpq*, *adelmrs*, *afghkrs*, *afglmpq*,

bdfhlpr, *bdfkmqs*, *beghlqs*, *begkmpr*, *cdghmps*, *cdgklqr*, *cefhmqr*, *cefklps*.

C^{12} D^3: *abcdefg*, *abchklm*, *abcpqrs*, *adehkpq*, *adelmrs*, *afghkrs*, *afglmpq*, *bdfhlpr*, *bdfkmqs*, *beghlqs*, *begkmpr*, *cdghmqr*, *cdgklps*, *cefhmps*, *cegklqr*. The sets containing *abc* are of type D.

B^8 C^6 D: *abcdefg*, *abchklm*, *abcpqrs*, *adehkpq*, *adelmrs*, *afghkrs*, *afglmpq*, *bdfhlpr*, *bdfkmqs*, *beghmps*, *begklqr*, *cdghmqr*, *cdgklps*, *cefhlqs*, *cefkmpr*. The set *abcdefg* is of type D, the other sets containing *a* are of type C.

B^{14} D: *abcedfg*, *abchklm*, *abcpqrs*, *adehkpq*, *adelmrs*, *afghlpr*, *afgkmqs*, *bdfhkrs*, *bdflmpq*, *beghmqr*, *begklps*, *cdghlqs*, *cdgkmpr*, *cefhmps*, *cefklqr*. The set *abcedfg* is of type D.

A^8 C^7: *abcdefg*, *abchklm*, *abcpqrs*, *adehkpq*, *adelmrs*, *afghkrs*, *afglmpq*, *bdfhlpr*, *bdgklqs*, *befkmps*, *beghmqr*, *cdfhmqs*, *cdgkmpr*, *cefklqr*, *ceghlps*. The sets containing *a* are of type C.

Let us now consider the groups of permutations of the fifteen letters under which these arrangements remain invariant. The type D^{15} is the one previously discussed, invariant under a group of order 20160.

Consider the type C^{12} D^3. The group of automorphisms of this arrangement permutes among themselves the three sets D and the twelve sets C; there is thus a sub-group of index three keeping fixed the set *abcdefg*. This sub-group effects among the seven letters *abcdefg* a group of permutations which keep fixed the triad *abc* and permute the triads *ade*, *afg*, *bdf*, *beg*, *cdg*, *cef* among themselves; it is thus of order 24. Finally, if each of the letters in the set *abcdefg* be fixed, any permutation of the group of automorphisms must belong to the Abelian group of order 8 generated by

$$(hk)\ (lm)\ (pq)\ (rs),\ (hl)\ (km)\ (pr)\ (qs),\ (hp)\ (kq)\ (lr)\ (ms).$$

The whole group of automorphisms is thus of order 3.24.8 or 576, generated by the permutations just written together with

$$(bc)\ (de)\ (lm)\ (rs),\ (de)\ (fg)\ (pq)\ (rs),\ (df)\ (eg)\ (hl)\ (km),$$
$$(abc)\ (efg)\ (klm)\ (qrs),\ (dhp)\ (ekq)\ (flr)\ (gms).$$

Next, consider the type $B^8\ C^6\ D$. Any operation of the group must keep fixed the set *abcdefg*, which is of type D. If each letter in this set is fixed the permutation must belong to the G_8 generated by

$$(hk)\ (lm)\ (pq)\ (rs),\ (hl)\ (km)\ (pr)\ (qs),\ (hp)\ (kq)\ (lr)\ (ms).$$

The permutations which do not keep fixed all these symbols must keep fixed the symbol *a*, which occurs in all the sets of type C, and must permute the sets *abc, ade, afg, bdf, beg, cdg, cef* among themselves. The permutations of *abcdefg* which do this form a group of order 24 generated by $(de)\ (fg)$, $(df)\ (eg)$, $(bdf)\ (ceg)$. The second of these permutations, however, does not correspond to an operation of the group of automorphisms of the arrangement. The latter is thus of order 96, generated by the operations of G_8 together with

$$(de)\ (fg)\ (hk)\ (pq), \qquad (bdf)\ (ceg)\ (lpr)\ (mqs).$$

If we denote the eight sets of type B by the digits $1, \ldots, 8$, this group is represented as a transitive group on the eight digits, the five operations generating it being respectively

$$(12)\ (34)\ (56)\ (78),\ (34)\ (56),\ (56)\ (78),\ (14)\ (23)\ (58)\ (67),$$
$$(357)\ (468).$$

Now the groups of order 96 and degree 8 have been examined by Burns, and the one here considered is type 8 on her list.[8] For this latter is the group generated by the permutations $(ACH)\ (BDG)$, $(ADE)\ (BCF)$, and the positive combinations of $(AB)\ (CD)\ (EF)\ (GH)$. Denoting the five generators of our group respectively by S_1, S_2, S_3, T, U, we have $US_2 = (367)\ (458)$ and $TUT = (175)\ (286)$. If we put now

$$(12345678) = (GHEFBADC),$$

we get Burns's operations. The identity of the two groups follows, since they have the same order and one contains the generators of the other.

[8] Burns, 3, 208.

Any automorphism of the type B^{14} D keeps fixed the set *abcdefg*. If every letter of this set is fixed, the operations belong to the G_8 which has occurred in connection with the other types. Of the N automorphisms, $N/7$ keep a fixed, these permute the pairs *bc*, *de*, *fg* among themselves, and $N/21$ keep the pair *bc* fixed as well. It is not difficult to see that such an automorphism must fix individually each of the letters b, c, d, e, f, g. Thus our group has order 21.8 or 168.

An automorphism of an arrangement of the type A^8 C^7 keeps fixed the letter a and permutes the seven sets C among themselves. A sub-group of index 7 keeps fixed one of these sets, say *abcdefg*. If all the letters of this set are fixed the permutation must be identity. Any permutation of *bcdefg* must keep fixed the pairs (*bc*, *de*, *fg*) or permute them cyclicly. There are 24 such permutations, so that the order of the entire group is once again 168.

Since $15 = 2.7 + 1$, we can use the method of §4 to obtain solutions in this case from the arrangement of seven symbols in triads. We take two sets of seven symbols, $a_1 \ldots a_7$, $b_1 \ldots b_7$, arranging them in triads $A_1, \ldots, A_7, B_1, \ldots, B_7$. The fifteen sets formed with the letters a, b, and an extra letter c are then, in the notation of §4, $a_1 \ldots a_7$, cA_iB_i, and $A_iB'_i$. Evidently the set $a_1 \ldots a_7$ in this arrangement is of type D, since the other sets have in common with it the seven sets A_i, each coming in twice. Conversely, if we have an arrangement in which one of the sets is of type D, it can be obtained in this way, taking for the letter c any of the eight letters which do not occur in the set.

If N is the order of the group of automorphisms of the arrangement and ρ the number of sets of type D, then any one of these sets, $a_1 \ldots a_7$ say, is invariant under a sub-group of index ρ, and a sub-group of index eight in this keeps fixed one of the remaining symbols, say c. The operations of this group permute the seven sets (A_iB_i) among themselves.

Each operation of the G_{168} which, operating on $a_1 \ldots a_7$, keeps

the set of triads $A_1 \ldots A_7$ fixed, permutes these triads among themselves. The G_{168} thus defines a group Γ_a of permutations of the sets (A_iB_i), each operation of which corresponds to a permutation of $a_1 \ldots a_7$ which leaves the triads $A_1, \ldots, A_7$ unaltered. There is similarly another group Γ_b of permutations of these sets, each operation of which corresponds to a permutation of $b_1 \ldots b_7$ leaving the triads $B_1, \ldots, B_7$ unaltered. The groups Γ_a and Γ_b are conjugate in the group of order 7! of all permutations of the sets (A_iB_i). In fact we can set up a correspondence between the symbols a and b so that to each triad A_i of the former set corresponds a triad B_i^* of the latter set which is one of B_i. If S is the permutation changing $B_1^* \ldots B_7^*$ into $B_1 \ldots B_7$, then $\Gamma_b = S\Gamma_a S^{-1}$. Any operation common to both Γ_a and Γ_b corresponds to a simultaneous permutation of $a_1 \ldots a_7$ and $b_1 \ldots b_7$ which keeps fixed the sets of triads $A_1 \ldots A_7$, $B_1 \ldots B_7$ and permutes the sets (A_iB_i). Accordingly such an operation belongs to the group of order $N/8\rho$ mentioned above. The possible forms for this group are thus obtained by finding the sub-groups common to Γ_a and one of its conjugates in the symmetric group of degree seven. There are four possibilities in all; the groups Γ_a and Γ_b may be identical, or may have in common an octahedral sub-group, a tetrahedral sub-group, or a sub-group of order 21. The corresponding values of ρ are respectively 15, 3, 1, 1 and so we verify the previous results for the value of N.

6.[9] It is remarkable that, apart from the trivial case when $n = 1$, all the operations of our groups are even permutations of the $4n - 1$ objects. We shall prove this by considering each type of group.

Let $\xi_1, \xi_2, \ldots, \xi_k$ be the "coordinates" considered in §2. It is known[10] that the corresponding group $LF(k,2)$ is generated by operations of the form

$$\xi_r' = \xi_r + \xi_s,\ \xi_i' = \xi_i\ (i = 1, \ldots, k;\ i \neq r, r \neq s).$$

[9] I owe this section to Dr. Coxeter.
[10] Dickson, 3, 78.

The only points of $PG(k-1,2)$ affected by this typical operation are those for which $\xi_s = 1$. These 2^{k-1} points are permuted in pairs. Regarded as a permutation of the 2^k-1 points, therefore, the operation consists of 2^{k-2} transpositions, and so is an even permutation if $k > 2$.

Again, the operations S_1, S_2, S_3 of §3 are even permutations of $0,1,2,\ldots,9,t$; and the operations

$$X' = X + 1 \quad \text{and} \quad X' = a^2X$$

(which generate the group for $p = 19$ or 23, and probably also for higher prime values) are even permutations of $0,1,\ldots,p-1$. Further, the operations

$$X' = X + 1, X' = X + x, \ldots, X' = X + x^{h-1} \quad \text{and}$$

$$X' = a^2X$$

(a a primitive root of $a^{p^h-1} = 1$) which generate the group when $p^h = 3^3$ and probably also in higher cases, are even permutations of the marks of $GF[p^h]$.[11]

Finally, it is clear that all the generating operations considered in §5 are even permutations of $a, b, \ldots, s$.

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[11] Dickson 3, 14.

REGULAR COMPOUND POLYTOPES IN MORE THAN FOUR DIMENSIONS

By H. S. M. Coxeter

1. *Introduction.* In this paper we consider cases when the vertices or bounding spaces of a regular polytope can be regarded as belonging to several repetitions of a simpler regular polytope. Kepler[1] observed that four of the eight vertices of the cube belong to a tetrahedron, and that eight of the twenty vertices of the dodecahedron belong to a cube. These two discoveries lead at once to a pair of tetrahedra whose vertices and bounding planes belong to a cube and an octahedron respectively, and to a set of five cubes whose vertices coincide in pairs with the vertices of a dodecahedron. Schoute[2] described some analogous "compounds" in four dimensions; and several more can be derived by combining these. (E.g., from a compound of polytopes Π having the vertices of one polytope π_1, and a compound of π_1's having the vertices of π_2, we deduce a compound of Π's having the vertices of π_2.)

It seems that the possibility of regular compounds in more than four dimensions has not been considered hitherto. The facts as developed here are somewhat surprising, especially the magnitude of the number of components that most compounds have. The subject is still left incomplete: Mr. Paley's theorems on orthogonal matrices[3] enable us to prove that compounds of simplexes whose vertices belong to a measure-polytope occur in

$$3, 7, 11, 15, \ldots, 87$$

dimensions, and that compounds of cross-polytopes whose vertices belong to a measure-polytope occur in

$$4, 8, 12, 16, \ldots, 88$$

[1] Kepler, 3, 271.
[2] Schoute, 5, 215, 216, 231.
[3] Paley, 4, 317.

dimensions; but the existence of such compounds in 91 and 92 dimensions remains uncertain.

Our position is stronger in the simple case when the vertices of the components are distinct and belong to one regular polytope, while the bounding spaces also are distinct and belong to another regular polytope. Of such compounds there are two in three and four dimensions, and, after that, just one whenever the number of dimensions is of the form $2^k - 1$.

Dr. Todd's results[4] enable us to find the number of components in certain cases.

2. *Definitions.* A compound polytope in m dimensions is defined to be a set of D equivalent and concentric m-dimensional regular polytopes Π. By saying that these polytopes are equivalent, we mean that they all stand in the same relation to the whole set, i.e. that any one of them can be transformed into any other by a symmetry of the set. In the present paper we restrict Π to be finite and convex. Let V and F be the numbers of its vertices and bounding figures.

The compound polytope is said to be *face-regular* if the DF bounding spaces of its components consist of the f bounding spaces of another regular polytope π, each taken d times ($d \geqslant 1$). Here

$$DF = df.$$

The d bounding figures (of various Π's) that lie in one bounding space of π are said to constitute the *bounding figure* of the compound polytope. This is simple or compound according as $d = 1$ or $d > 1$.

The compound polytope is said to be *vertex-regular* if the DV vertices of its components consist of the v vertices of another regular polytope π_1, each taken d_1 times ($d_1 \geqslant 1$). Here

$$DV = d_1 v.$$

The *vertex figure*[5] of the compound polytope can be defined as the section by an $(m - 1)$-space which passes through the fur-

[4] Todd, 6.
[5] Coxeter, 1, 336.

ther ends of a set of covertical edges. This is simple or compound according as $d_1 = 1$ or $d_1 > 1$.

A compound polytope which is both face-regular and vertex-regular is said to be *primary* if its bounding figure and vertex figure are both face-regular and vertex-regular (e.g., if $d = d_1 = 1$, as in the case described near the end of Section 1); otherwise it is said to be *secondary*. Finally, a compound polytope is said to be *tertiary* if it is face-regular but not vertex-regular, or *vice versa*.

We may denote primary or secondary compounds by the symbol

$$d_1\pi_1[D\Pi]d\pi,$$

and tertiary compounds by $[D\Pi]d\pi$ or $d_1\pi_1[D\Pi]$. If Π', π', π_1' are the reciprocals of Π, π, π_1, then the reciprocals of the above compounds are easily seen to be

$$d\pi'[D\Pi']d_1\pi_1', \quad d\pi'[D\Pi'] \quad \text{and} \quad [D\Pi']d_1\pi_1',$$

respectively.

3.[6] *Regular Compounds in Two, Three and Four Dimensions.* Here we shall be content to state results without proof, as our main object is the investigation of higher spaces. In Table 1, compounds placed side by side are reciprocal; therefore any compound having a line to itself is self-reciprocal, although $\{5,3\}$ $[5\ \alpha_3]$ $\{3,5\}$[7] might be more accurately described as *enantiomorpho-reciprocal*.

It is remarkable that the vertices of $\{5,3,3\}$ include those of all the other regular polytopes in four dimensions. There are two distinct compounds of $\{3,4,3\}$'s having the vertices of $\{5,3,3\}$; the 25 do not occur among the 200. Similarly for the compounds of β_4's and of γ_4's that have the vertices of $\{5,3,3\}$ and the bounding spaces of $\{3,3,5\}$.

[6] This section can be omitted without impairing the rest of the paper.

[7] α_3 denotes the tetrahedron, $\{5,3\}$ the dodecahedron, $\{3,5\}$ the icosahedron. See Coxeter, 1, 342–344.

TABLE 1

Number of dimensions	Regularity	Compound Polytopes
2	Primary	$\{Dk\}\,[D\{k\}\,]\,\{Dk\}$
3	Primary	$\gamma_3[2\alpha_3]\beta_3$
	Primary	$\{5,3\}\,[5\alpha_3]\,\{3,5\}$
	Secondary	$2\{5,3\}\,[10\alpha_3]2\{3,5\}$
	Tertiary	$[5\beta_3]2\{3,5\},\quad 2\{5,3\}\,[5\gamma_3]$
4	Primary*	$\{5,3,3\}\,[120\alpha_4]\,\{3,3,5\}$
	" †	$\{3,3,5\}\,[5\{3,4,3\}\,]\,\{5,3,3\}$
	" ‡	$\{3,4,3\}\,[3\beta_4]2\{3,4,3\},\quad 2\{3,4,3\}\,[3\gamma_4]\,\{3,4,3\}$
	"	$\{3,3,5\}\,[15\beta_4]2\{5,3,3\},\quad 2\{3,3,5\}\,[15\gamma_4]\,\{5,3,3\}$
	"	$\{5,3,3\}\,[75\beta_4]2\{3,3,5\},\quad 2\{5,3,3\}\,[75\gamma_4]\,\{3,3,5\}$
	Secondary†	$\{5,3,3\}\,[25\{3,4,3\}\,]5\{5,3,3\},$
		$5\{3,3,5\}\,[25\{3,4,3\}\,]\,\{3,3,5\}$
	"	$5\{3,3,5\}\,[75\beta_4]10\{5,3,3\},\quad 10\{3,3,5\}\,[75\gamma_4]5\{5,3,3\}$
	" •	$8\{5,3,3\}\,[600\beta_4]16\{3,3,5\},\quad 16\{5,3,3\}\,[600\gamma_4]8\{3,3,5\}$
	Tertiary§	$\gamma_4[2\beta_4],\quad [2\gamma_4]\beta_4$
	" ¶	$\{5,3,3\}\,[5\{3,3,5\}\,],\quad [5\{5,3,3\}\,]\,\{3,3,5\}$
	" ¶	$2\{5,3,3\}\,[10\{3,3,5\}\,],\quad [10\{5,3,3\}\,]2\{3,3,5\}$
	" •	$8\{5,3,3\}\,[200\{3,4,3\}\,],\quad [200\{3,4,3\}\,]8\{3,3,5\}$

4. *Possibilities in More Dimensions.* In more than four dimensions the possibilities of compounds are substantially reduced, since the only finite regular polytopes are the simplex α_m, the cross-polytope β_m, and the measure-polytope γ_m. (γ_m is the cube-analogue, and β_m its reciprocal.)

In a compound of the form $d_1\pi_1[D\Pi]$, π_1 must be γ_m, since α_m and β_m have too few vertices. In fact, for each vertex of α_m there is an opposite bounding α_{m-1} possessing all the other vertices, while for each pair of opposite vertices of β_m there is an

* These compounds cannot be deduced from Schoute's. They are given here for the sake of completeness.

† Schoute, 5, 231. (But surely *zehn* should be *fünf* in the phrase *auf zehn verschiedene Arten*.)

‡ Schoute, 5, 216.

§ Schoute, 5, 215.

¶ Schoute, 5, 231.

"equatorial" β_{m-1} possessing all the other vertices. Since π_1 is γ_m, Π must be either α_m or β_m. But if Π is α_m, so that D α_m's have the vertices of a γ_m, each of these vertices corresponds to an opposite bounding α_{m-1} of any α_m possessing it, and consequently the D α_m's have the bounding spaces of a β_m (reciprocal to the γ_m). Thus, writing d for d_1 since there is now no confusion, the only possible vertex-regular compounds are

$$d\gamma_m[D\alpha_m]d\beta_m \qquad \text{and} \qquad d\gamma_m[D\beta_m].$$

The face-regular compounds are the reciprocals of these, namely

$$d\gamma_m[D\alpha_m]d\beta_m \qquad \text{and} \qquad [D\gamma_m]d\beta_m.$$

To sum up, the only possibilities are: a self-reciprocal compound $d\gamma_m[D\alpha_m]d\beta_m$ which is primary or secondary according as $d = 1$ or $d > 1$, and reciprocal tertiary compounds $d\gamma_m[D\beta_m]$, $[D\gamma_m]d\beta_m$. We shall soon see that such compounds do not occur for all values of m.

5. *The Coexistence of* $d\gamma_{m-1}[D\alpha_{m-1}]d\beta_{m-1}$ *and* $d\gamma_m[D\beta_m]$. β_m and γ_m both have the property that a pair of opposite bounding figures possess together all the vertices. It follows that any bounding space of γ_m contains a bounding α_{m-1} of every component of $d\gamma_m[D\beta_m]$. Conversely, if we inscribe $d\gamma_{m-1}[D\alpha_{m-1}]d\beta_{m-1}$ in each of two opposite bounding γ_{m-1}'s of γ_m, then every α_{m-1} inscribed in the one γ_{m-1} can be joined to the oppositely situated α_{m-1} inscribed in the other, so as to make a β_m inscribed in γ_m. Hence $d\gamma_{m-1}[D\alpha_{m-1}]d\beta_{m-1}$ and $d\gamma_m[D\beta_m]$ (and so also $[D\gamma_m]d\beta_m$) exist for the same values of m, d and D.

Since α_{m-1} has m vertices while γ_{m-1} has 2^{m-1},

$$D/d = 2^{m-1}/m. \tag{5.1}$$

It follows that the primary compound $\gamma_{m-1}[D\alpha_{m-1}]\beta_{m-1}$ (with $d = 1$) is only possible if m is a power of 2. (We shall see later that m can in fact be any power of 2.)

6. *The Relation of Paley's U-matrices*[8] *to the Compounds of Cross-polytopes.* From a U-matrix of m rows and columns we

[8] Paley, 4, 312.

can construct another U-matrix by reversing all the signs. By putting these two matrices together we obtain a matrix of $2m$ rows and m columns. It is clear that the orthogonality is preserved except between pairs of rows which are of opposite sign throughout.

If we interpret each row as a point in m dimensions, the $2m$ rows represent the vertices of a β_m. Since the 2^m possible arrangements of m elements ± 1 represent the vertices of γ_m, we have obtained one component of the compound $d\gamma_m[D\beta_m]$.

Conversely, consider any β_m inscribed in γ_m. Its vertices occur in m opposite pairs. By selecting one vertex from each pair, we obtain m points whose directions from the origin are mutually perpendicular, so that their coördinates form a U-matrix.

7. *The Relation of Paley's U-matrices to the Compounds of Simplexes.* The corresponding component of $d\gamma_{m-1}[D\alpha_{m-1}]d\beta_{m-1}$ can be derived from the U-matrix as follows. By reversing all the signs of any row whose first element $(A_{i,0})$ is negative, we obtain a U-matrix with "+" throughout the first column. By omitting this column we derive a matrix of $m-1$ columns whose m rows represent the vertices of a regular simplex α_{m-1}. Since the 2^{m-1} possible arrangements of $m-1$ elements ± 1 represent the vertices of γ_{m-1}, we have obtained one component of the compound $d\gamma_{m-1}[D\alpha_{m-1}]d\beta_{m-1}$.

Conversely, from any α_{m-1} inscribed in γ_{m-1} we derive a U-matrix by appending a column of $+1$'s to the matrix of coördinates.

8. *Conclusion as to the Possible Values of m.* We now see that the values of m for which

$$d\gamma_{m-1}[D\alpha_{m-1}]d\beta_{m-1}, \quad d\gamma_m[D\beta_m] \quad \text{and} \quad [D\gamma_m]d\beta_m$$

exist are precisely the values of m for which U-matrices exist.

In particular, apart from the trival cases $m = 1, 2$ $(d = D = 1)$, m must be a multiple of 4.[9] It follows incidentally that there are no regular compound polytopes in five or six dimensions.

[9] Paley, 4, 311.

Paley's Lemma 4 shows that our compounds exist whenever m is divisible by 4 and of the form

$$2^k(p^k + 1),$$

where p is an odd prime. One is tempted to believe that compounds

(8.1) $d\gamma_{4n-1}[D\alpha_{4n-1}]d\beta_{4n-1}$, $d\gamma_{4n}[D\beta_{4n}]$ and $[D\gamma_{4n}]d\beta_{4n}$

exist for all values of n; but Paley's Lemma 6 shows that we are still uncertain about the cases

$$n = 23, 29, 39, 43, 46, 47 \ldots .$$

9. *Primary Compounds.* Of the 2^m arrays[10] B, 2^{m-1} have $B_0 = +1$. Thus Paley's $2^m/m$ U-matrices lead, in the manner described in Section 7, to $2^{m-1}/m$ separate α_{m-1}'s. Putting $m = 2^k$, we have the following theorem:

The primary compound

(9.1) $$\gamma_{2^k-1}[2^{2^k-k-1}\alpha_{2^k-1}]\beta_{2^k-1}$$

exists for all integers k.

Of course $\gamma_{2^k}[2^{2^k-k-1}\beta_{2^k}]$ and $[2^{2^k-k-1}\gamma_{2^k}]\beta_{2^k}$ exist too; but we do not call these primary, since the former is only vertex-regular and the latter only face-regular. For instance, although the vertices of $[2\ \gamma_4]\beta_4$ coincide with those of $\{3,4,3\}$, eight of them belong to both the γ_4's while the remaining sixteen belong to only one.

10. *Todd's Groups.* Todd's work[11] enables us to find possible values for d and D.

If N is the order of the group of automorphisms of one of his arrangements of sets of objects, the $(4n-1)!$ possible permutations of the objects lead to $(4n-1)!/N$ essentially distinct arrangements of sets. We now interpret the objects as Cartesian

[10] Paley, **4**, 319.
[11] Todd, **6**, 322–325, 329–331.

coordinates in $4n - 1$ dimensions. With each set we associate that point which has $+1$ for every coordinate corresponding to an object of the set and -1 for each of the remaining $2n$ coordinates. Todd's arrangement of sets thus leads to $4n - 1$ points which are the vertices of an α_{4n-2}, and which, when taken along with the special point $(+1, +1, \ldots, +1)$, form an α_{4n-1} inscribed in γ_{4n-1}.

It follows that compound polytopes (8.1) occur with

$$d = (4n - 1)!/N.$$

By (5.1),

$$D = 2^{4n-3}d/n.$$

11. *Conclusion as to the Possible Values of D.* In the "finite geometry" solution[12] for $m = 2^h$, the group is[13] $LF(k,2)$ so that

$$N = (2^h - 1)(2^h - 2)(2^h - 2^2)\ldots(2^h - 2^{h-1}) = 2^{\frac{1}{2}h(h-1)}F(k)$$

where[14]

$$F(k) = (2 - 1)(2^2 - 1)\ldots(2^k - 1).$$

Hence

$$d = (2^h - 1)!/N = (2^h - 3)!/2^{\frac{1}{2}(h-2)(h+1)}F(k - 2)$$

and

$$D = 2^{2^h - \frac{1}{2}h(h+1)}(2^h - 3)!/F(k - 2).$$

With the help of Todd's other results, we can make Table 2.

Our information is complete for $n \leqslant 4$. In particular, we know that just $64\ 561\ 751\ 654\ 400 = 2^7.3^4.13!$ α_{15}'s can be inscribed in γ_{15}.

The cases $n = 5, 6, 7$ suggest, as a conjecture, that whenever $n > 4$ and $4n - 1 = p^h$, the "quadratic residues" solution (in a Galois field, if $h > 1$) gives

[12] Paley, 4, 318. Todd, 6, 322.
[13] Dickson, 2, 87.
[14] $F(0) = 1$, $F(1) = 1$, $F(2) = 3$, $F(3) = 21$, $F(4) = 315$, $F(5) = 9765$.

$$N = (2n - 1)(4n - 1),$$

whence

$$d = 2(4n - 3)!$$

and

$$D = 2^{4n-3}(4n - 3)!/n.$$

TABLE 2

n	Type of Solution	Group	N	d	D
1	General	Symmetric (degree 3)	6	1	2
2	General	$LF(2,7)$	168	30	480
3	General	$LF(2,11)$	660	60,480	10,321,920
4	D^{15}	Alternating (degree 8)	20,160	64,864,800	132,843,110,400
	$C^{12}D^3$		576	2,270,268,000	4,649,508,864,000
	B^3C^9D		96	13,621,608,000	27,897,053,184,000
	$B^{14}D$		168	7,783,776,000	15,941,173,248,000
	A^8C^7		168	7,783,776,000	15,941,173,248,000
5	"Quadratic residues"		171	2.17!	$2^{13}.17!/5$
	(Others)		?		
6	"Quadratic residues"		253	2.21!	$2^{21}.7.20!$
	(Others)		?		
7	"Galois field"		351	2.25!	$2^{28}.25!/7$
	(Others)		?		
8	"Finite geometry"	$LF(5,2)$	9,999,360	$29!/2^9.21$	$2^{17}.29!/21$
	(Others)		?		

12. *The Possibility of Halving D.* The components of the primary compound (9.1) occur among those of the $d\gamma_{2k-1}$ $[Da_{2k-1}]$ $d\beta_{2k-1}$ considered at the beginning of Section 11. This fact suggests that there may be other cases in which a subset of

the components of a given compound can be chosen in such a symmetrical way as to form a compound by themselves. The only cases that we shall consider here are those in which the subset consists of half the total number of components. The simplest example is the splitting of $2\{5,3\}$ $[10\ \alpha_3]$ $2\{3,5\}$ into two (enantiomorphous) $\{5,3\}$ $[5\ \alpha_3]$ $\{3,5\}$'s.

It is perhaps worth while to remark that, in any compound of the form $d\gamma_{4n-1}[D\alpha_{4n-1}]d\beta_{4n-1}$, all the vertices of $\frac{1}{2}D$ of the α_{4n-1}'s belong to the uniform polytope[15] $h\gamma_{4n-1}$. (For, if one vertex of a component α_{4n-1} has $4n-1$ coordinates $+1$, the remaining vertices have $2n$ coordinates -1, and $2n$ is even.) Similarly, in any compound of the form $d\gamma_{4n}[D\beta_{4n}]$, all the vertices of $\frac{1}{2}D$ β_{4n}'s belong to $h\gamma_{4n}$. But, since $h\gamma_{4n-1}$ and $h\gamma_{4n}$ are not regular for $n > 1$, the new compounds so derived are neither vertex-regular nor face-regular.

13. *Proof That d and D can be Halved in All the Cases Considered in Section 11 (except, of course, in the case when $d = 1$).* Consider the γ_{4n-1} whose vertices have coordinates $(\pm 1, \pm 1, \ldots, \pm 1)$. Its $2^{4n-1}(4n-1)!$ symmetries consist of the 2^{4n-1} possible changes of sign of the coordinates, combined with the $(4n-1)!$ permutations. In half these symmetries, the permutation will be even; we shall call these *even*[16] symmetries.

Corresponding to any one of Todd's groups, we have a compound $d\gamma_{4n-1}[D\alpha_{4n-1}]d\beta_{4n-1}$. Whenever it happens that all operations of the group are even permutations of the coordinates, the totality of even permutations will give rise to $\frac{1}{2}d$ of the d α_{4n-1}'s at one vertex, and the totality of "even symmetries" of γ_{4n-1} will give rise to the compound

$$\tfrac{1}{2}d\gamma_{4n-1}[\tfrac{1}{2}D\alpha_{4n-1}]\ \tfrac{1}{2}d\beta_{4n-1}.$$

By Todd's §6, this does actually happen in all the cases that have been investigated. For instance, $30\gamma_7[480\alpha_7]30\beta_7$ leads to $15\gamma_7[240\alpha_7]15\beta_7$.

[15] Coxeter, 1, 363.

[16] They must not be confused with the positive (or rotational) symmetries, of which there are the same number.

Of course the corresponding compounds in $4n$ dimensions occur too.

14. *Recapitulation.* To sum up, all regular compound polytopes in more than four dimensions are of the form

$$d\gamma_{4n-1}[D\alpha_{4n-1}]d\beta_{4n-1}, \quad d\gamma_{4n}[D\beta_{4n}] \quad \text{or} \quad [D\gamma_{4n}]d\beta_{4n}.$$

It is conjectured that such compounds exist for all values of n, there being only six doubtful cases for $n \leqslant 50$. It is known that they exist whenever $4n$ is of the form $2^k(p^k + 1)$, where p is an odd prime.

d and D are connected by the relation

$$D/d = 2^{4n-3}/n.$$

d can be unity whenever n is a power of 2. If $n = 2^{k-2}(> 1)$, we can also have

$$d \text{ or } \tfrac{1}{2}d = (2^k - 3)!/2^{\frac{1}{2}k(k-1)}.1.3.7 \ldots (2^{k-2} - 1).$$

If $n = 1$, we can only have $d = 1$.
If $n = 2$, we can have $d = 1$, 15 or 30.
If $n = 3$, we can have $d = 30{,}240$ or $60{,}480$.
If $n = 4$, we can have

$$d = 1, \quad 32{,}432{,}400 \text{ or } 64{,}864{,}800,$$
$$1{,}135{,}134{,}000 \text{ or } 2{,}270{,}268{,}000,$$
$$6{,}810{,}804{,}000 \text{ or } 13{,}621{,}608{,}000,$$

and, in two distinct ways, $3{,}891{,}888{,}000$ or $7{,}783{,}776{,}000$.

Finally, we can have

$$d = (4n - 3)! \quad \text{or} \quad 2(4n - 3)!$$

if $n = 5, 6, 7$, and probably also in higher cases when $4n - 1$ is a prime power. But we should expect to find a great variety of other possible values of d for each value of n (> 4).

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